An Anthology of
Problems in Mathematics

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50 Problems at the graduate level
in Geometry, Algebra,Analysis
Linear Algebra, Logic. Dynamical Systems,
Physics,and Number Theory
Part II
Problems 28-50

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logic
An Algebra for Possibility

To clarify the following construction , it is helpful to keep in mind a pair of sets A and B, in the plane, such that

$$A \cap B = C \neq \varnothing$$

W is a "Possibility Algebra" with 3 values T (true) , F (false) and P (possible) . We notate an expression such as "x may be in B" as x ~ B ; it has the value "P" .

In this logic "Possibly" and "Possibly Not" are deemed equivalent. Therefore $\boxed{P\neg = \neg\neg P = P}$.

"Not Possible" $\boxed{\neg P}$ **however is the same as "Impossible", therefore False (F) . The truth table for AND ($\wedge$) will be constructed by reference to the Venn Diagram:**

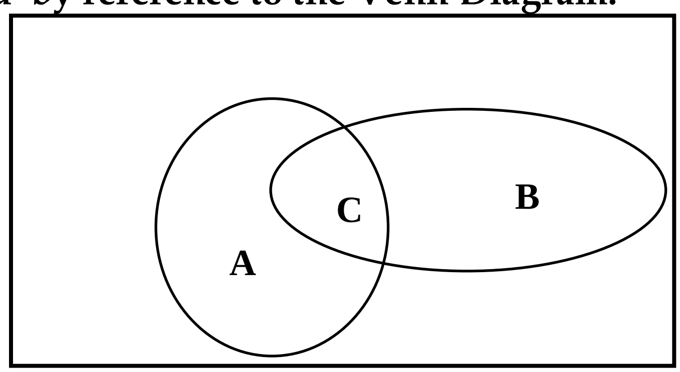

Figure 4

For example: The statement "If x *is* in A and x *may be* in B, then x *may be* in C" is represented as

$$[x \in A \wedge x \sim B \rightarrow x \in C] = "P";$$
$$T \wedge P = P$$

The statement " If x *is not* in A and x *may be* in B, then x *will not be* in C " becomes :

$$[x \notin A \wedge x \sim B \rightarrow x \notin C] = "F";$$
$$F \wedge P = F \quad .$$

Problem 28:

(i) Using these examples as a guide, construct the complete table in W for " $\wedge$ " .

(ii) Construct the truth table for " $\vee$ " based on the formula:

$$X \vee Y \equiv \neg(\neg X \wedge \neg Y)$$

(iii) Verify that $\wedge$ and $\vee$ are commutative, associative and reflexive.

Problem 29:

(iv) Because of (iii), W can be modeled by 3x3 matrices. Compute these from the table for $\wedge$.

(v) Show that, in fact, one can model $\wedge$ from the matrix product XY , and $\vee$ from the function X+Y - XY

(vi) Show that there exists no modular arithmetic Z_k in which W may be embedded as a Boolean Algebra.

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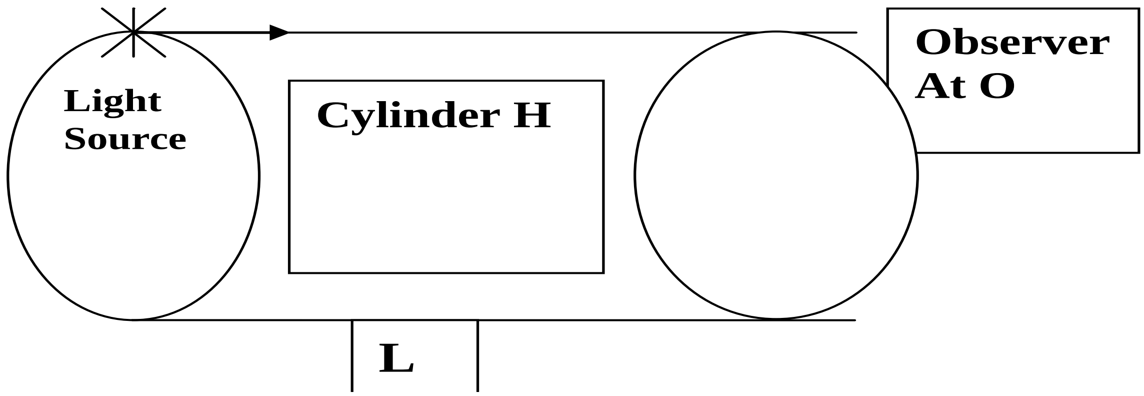
Physics

Light Intensity on a Cylinder

Let H be a cylinder of infinite length, with its base centered on the origin, extending to the left on the x-axis, and with circumference C = 1.

A source of light is placed on the upper generator line z=0, at distance L from the origin. At the origin, on this same generator, there is an observer O.



The intensity at the light source is I_0. Light rays are constrained to move on the surface of the cylinder, the intensity of a beam of light along a geodesic diminishing as the inverse square of the distance from the source.

Problem 30:

(i) Calculate the total intensity received by the observer O.

(ii) Show that, at great distances, the intensity diminishes (asymptotically) as O(1/L), the inverse of the length.

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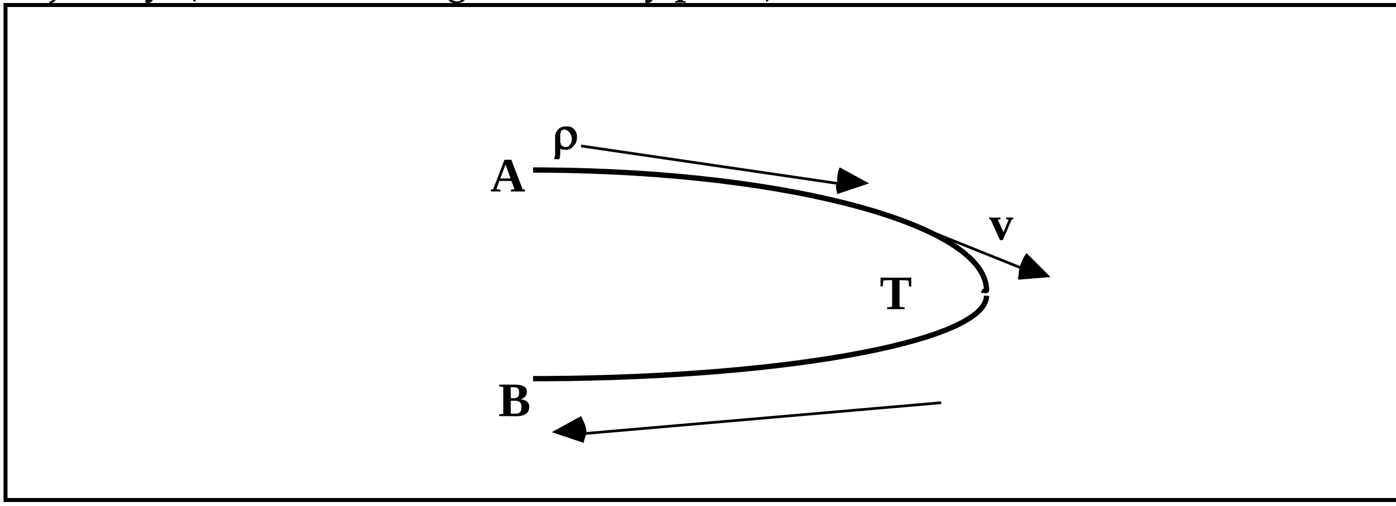
Relativity Problems

Prefatory Note on Special Relativity : **Imagine that a rocket ship ρ is traveling from planet A to planet B along some sort of smooth trajectory, (one with a tangent at every point)**



If the velocity of ρ *along the tangent* **at every point is a uniform v , then the** *proper time* **, that is to say, the passage of time** *as experienced by the crew on the ship* **, will be a function only of the** *total length* **D of the trajectory T,** *and independent of its shape* **. In terms of the** *local time* **t on planets A and B, assumed to be at rest relative to one another , and the tangential velocity v, the** *proper time* **is given by :**

$$\boxed{\tau = \beta t}\text{, where}$$

$$\boxed{\beta = \sqrt{1 - v^2/c^2}}$$

Since time = distance/velocity , one has:

$$\boxed{t = \frac{D}{v};\ \tau = \frac{D\sqrt{1 - v^2/c^2}}{v}}$$

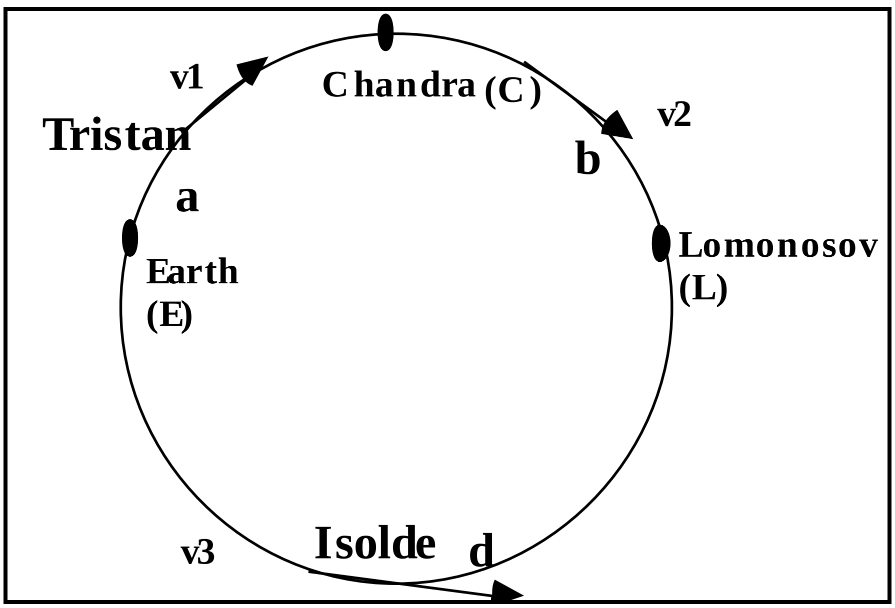
It may be helpful in what follows to work with the quantities

$$u_j = {}^{v_j}\!/_{c}\ ; j = 1,2,3$$.

However using the actual velocities and the speed of light in computations makes it easier, particularly in a complicated equation, to detect errors.

Problem 31:

Time Travel Along Circular Paths



Two spaceships, the *Tristan* and the *Isolde* , begin their journeys from the earth to other planets in distant galaxies at the same time. They move along the circumference of the exscribed circle about the triangle formed by planets Earth ,Chandra and Lomonosov.

Arc-length between E and C = a

Arc-length between C and L = b

Arc-length between E an d L = d

The Tristan and the Isolde leave Cape Canaveral at time $t = 0$

Isolde travels along the lower arc of the circle with a uniform, and given, velocity v_3 .

Tristan moves from Earth to Chandra with unknown velocity v_1 ; then from Chandra to Lomonosov with unknown velocity v_2 .

Assume that the Tristan and the Isolde:

(α) arrive at Lomonosov at the same *local time* $t_3 = t_1 + t_2$;

(β) arrive at Lomonosov in the same *proper time* $\tau_3 = \tau_1 + \tau_2$.

(That is to say, their crews have aged by the same amount.)

(i) Show that $d \geq a+b$

(ii) The quantities

$$\boxed{\begin{aligned} h_1 &= \frac{d^2 + a^2 - b^2}{2ad} \\ h_2 &= \frac{d^2 + b^2 - a^2}{2ad} \end{aligned}}$$

play an important role in these problems.

Show that(i) implies $\boxed{h_1 \geq 1 \wedge h_2 \geq 1}$

(iii) Under the assumption that (i) is satisfied, solve for u_1 and u_2 .

(iv) Under the assumption that (i) is satisfied, show that, with the exception of certain singular values of u_3 , there will be two distinct solution sets (u_1, u_2) and (u_1' , u_2')

(v) For a given value of $v_3 < c$, there are singular values of h_1 and h_2 when there is a special relationship between lengths a and b. For these cases there is no solution. What are they?

(vi) Use a *topological argument* to show that whenever d < a+b the crew of the Tristan will age less than of the Isolde.

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Number Theory

Basis representations of unity

We will be looking at representations of real numbers where the base b = 1/x , x lying in the range $1/2 \leq x \leq 1$.

Let A = (a_1, a_2 ,...,a_n,...) be an infinite sequence of 0's and 1's , (not all 0's) . Let x be in the range specified above. If

$$(A,x) \equiv \sum_{i=1}^{\infty} a_i x^i = 1,$$

we will say that A is a unitary representation of x . Roughly speaking, unitary representations are adjoint or dual to binary decimal representations.

Problem 32:

(i) Given A , there is a unique positive x such that (A.x) is unitary.

(ii) For a given x in the specified range, there is at least one sequence A such that (A,x) is unitary.

Problem 33:

(i) If $\frac{\sqrt{5}-1}{2} < x < 1$, then there are at least two distinct sequences such that (A,x) and (B,x) are unitary

(ii) If $\frac{1}{2} < x \le \frac{\sqrt{5}-1}{2}$, and there is a finite polynomial:

$$P(x) = \sum_{i=1}^{j} a_i x^i = 1, a_i = 0 \; or \; 1$$

then there are at least 2 distinct sequences A and B such that (A.x) and (B,x) are unitary.

Problem 34:

Use the above to prove that for *all* x in the specified range, except at 1 and 1/2 , there are always several sequences

A_1, A_2 ,.... such that (A_i , x) = 1

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Diophantine Equations over Arithmetic Progressions

Problem 35 (Easy):

Let M(a,b) = { an +b} , a,b, integers, n = 0, ±1, ± 2 ....., be an arithmetic progression, such that

(a) b is not 0 , a > 0

(b) greatest common divisor(a,b) = (a,b)=1 .

The progression Z = (...,-2,-1,0,1,2 ,....) of all the integers will be called the 'trivial progression'. Clearly Z is always generated when a = 1.

Let p_n = an + b be a typical element of M . Show that if there are 3 integers l, m, n such that $\boxed{p_n = p_m + p_l}$, then M is the trivial progression.

(ii) Show that if there are 3 integers l, m, n such that

$$\boxed{p_n^2 = p_m^2 + p_l^2}$$

then M is the trivial progression. Note that the above relation is homogeneous, that is to say, if a,b is a solution, then ta, tb is also one where t is any integer. Thus, there are no Pythagorean triples in any non-trivial progression.

(iii) Show that when x =y+z , the solutions of $\boxed{xp_n = yp_m + zp_l}$, do not depend on a and b.

Problem 36 (More challenging):

We now look at the *modified Pythagorean equation* :

$$\text{(A)}: \quad \boxed{p_{n_1}^2 - p_{n_1} p_{n_2} + p_{n_2}^2 = p_{n_3}^2}$$

For this problem a and b do not have to be relatively prime.

Show that there exist solution sets $\boxed{(n_1, n_2, n_3)}$ for (A) which do not depend on a or b.

Problem 37 (Difficult):

Let $\boxed{a = 12r; b = 12s; (r,s) = 1}$.

(i) Show that (A) can be solved for all r, s.

(ii) Use this result to show that (A) can be solved for all integer pairs (a,b), b ≠ 0 .

Problem 38:

Give explicit formulae for the solution sets { n_1 ,n_2 ,n_3 }, in terms of independent parameters.

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Integer points on graphs parametrized by rational integral polynomials

Let

$$\boxed{\begin{array}{l} P(x) = a_0 x^e + a_1 x^{e-1} + \ldots + a_e \\ Q(x) = b_0 x^f + b_1 x^{f-1} + \ldots + b_f \end{array}}$$

{a_i }, {b_j } integers, a_0 , b_0 >0 . The range of x will be over the integers, Z .

Let G(n) = (P(n) , Q(n)) be the ***greatest common divisor*** **of P(n) and Q(n), and F(P,Q) = { P ,Q } be the** ***largest common algebraic factor*** **of P(x) and Q(x) . By convention we let g.c.d. (n,0) = n for all n .**

Problem 39:

(i) Prove that , when F(P,Q) = 1, the range of G(n) is finite.

(ii) Prove that when F(P, Q)=1, G(n) is periodic over Z.

(iii) Define $\boxed{P(x) = ax^2 + bx}$, a, b integers, (a,b) = 1, a>1. Find the set of all fractions r = p/q , (p,q)= 1 , q>1 , such that P(r) is an integer.

Problem 40 (challenging):

Let $\boxed{P(x) = ax^2 + 2bx}$ where

(i) a,b are non-zero integers, (a,b) = 1 . In particular

(ii) a is a prime >2

Establish the conditions on a and b for which there are *complex rationals* $\boxed{r = \frac{u+iv}{w}}$, (u,v,w integers , g.c.d(u,v,w)=1, v,w ≠ 0) , such that P(r) = -n is an integer. (It is clear that the right hand side must be a negative number) . Characterize the sets of values (u,v,w).

Problem 41 (very difficult):

Let $\boxed{P(x) = a_0 x^3 + 2a_1 x^2 - a_2 x}$;

(i) All coefficients are non-zero integers ;

(ii) a_0 >2 ;

(iii) $\boxed{(a_0, 2a_1) = 1}$

Find the general solution of the equation P(r) = n = integer , where r is a rational = p/q , (p, q)= 1, q >1.

Diophantine Equations With Square Roots

Problem 42:

Derive general formulae from which one can compute all the solutions to the Diophantine equation:

$$\textbf{(A):}\quad (a+\sqrt{d})^3 + (a-\sqrt{d})^3 = b^3$$

Here a, b and d are integers which may be positive or negative, but not 0. However, observe that Fermat's Last Theorem precludes non-trivial solutions when d is a perfect square! If d is negative then one has a pair of complex conjugates.

Problem 43 (somewhat more difficult):

Observe that if (a,b) is a solution, then so is (-a,-b) . Consider the specific equation:

$$\textbf{(B)}\quad \boxed{(a+6i\sqrt{2})^3 + (a-6i\sqrt{2})^3 = b^3}$$

Find at least two distinct solution sets ,

$$\boxed{\begin{array}{l}(i)\,(a_1,b_1),(-a_1,-b_1);\\ (ii)\,(a_2,b_2),(-a_2,-b_2)\end{array}}$$

Problem 44

(i) Show that all solutions of (B) must be divisible by 12.

(ii) It is easily seen that a must divide b^3 . Let

$$\boxed{u=\frac{a}{12};\,v=\frac{b}{12}}.$$

Substitute in equation (A) to derive a new equation in u and v. Observe that it is still the case that u divides v^3 .

Define quantities D and e , by

$$\boxed{D = \frac{u^3}{v} = 3^e Q}$$

where e is the highest exponent of 3 in D, Q the product of the remaining factors. Show that:

(i) There are solutions to Problem 14 if e = 0

(ii) There are no solutions for e $\geq$ 2 .

Problem 45 (much too difficult!) :

What about e = 1? Spend some time working on this before looking up the comments in the solutions section.

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The "Corkscrew" Equation

Define g(x,y,z) by:

$$\boxed{(i)\ g(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}}$$

and consider the solution sets of :

$$\boxed{(ii)\ g(x,y,z) = n}$$

for each n. The domain of g consists of the non-zero integer triples

(x, y, z) such that the greatest common divisor of (x,y,z) = gcd (x,y,z) = 1

Problem 46:

(a) Assume x, y, z > 0 .

(b) Assume z = 1

Find all integers n for which there are integer solutions. List those solutions. Prove that these are the only solutions under these conditions.

Problem 47:

Once again set z = 1, however this time x and y may be positive or negative (but obviously not 0) . Let r = p/q , a rational number in lowest terms, r>0 , q $\geq$ 2 , gcd (p,q) = 1

Fix q . Show that , for fixed denominator q, there are only finitely many solutions (x, y, z, p) of

$$\boxed{g(x, y, z) = r = p/q}$$

Problem 48:

When x < 0 , y, z> 0 , show that there are infinitely many integers m >0 for which there are integral solutions of: there are

$$\boxed{g(x, y, z) = -m}$$

Problem 40:

Show that the equation

$$\boxed{g(x, y, z) = n}$$

can always be transformed into

$$\boxed{\frac{a^3 + b^3 + c^3}{abc} = n},$$

where

$$\boxed{x = a^2 b, y = b^2 c, z = c^2 a}$$

Note: **This problem is not as easy as it may appear at first glance.**

Problem 50:

Let x, y, z, n > 0 . Using the result of problem 4, find at least 8 integral solutions of:

$$\boxed{g(x, y, z) = n}$$

Commentary : **After working on this set of problems, look in the solutions section to read the comments of number theorists Noam Elkies and Edray Herber Goins**