Solutions II

Problems 28-50

Problem 28:

(i), (ii) : The tables for *and* and *or* in the Possibility Logic W , are :

$\wedge$	T	F	P
T	T	F	P
F	F	F	F
P	P	F	P

$\vee$	T	F	P
T	T	T	T
F	T	F	P
P	T	P	P

(iii) Inspection shows that they are commutative and reflexive. Associativity follows from the associativity of the normal set theory operations.

Problem 29:

(iv) The matrix representation is:

$$T \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; F \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} ; P \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(v) The crucial observation here is that W has 3 idempotents for both its operations , and no Boolean Algebra

with 3 idempotents can be embedded into a modular arithmetic Z_k for any k .

If a is a typical idempotent element of W , then one wants

$$a \wedge a \equiv a \bullet a = (a)^2 \equiv a(\mathrm{mod}\, k)$$

Since $\neg P \equiv P$, **there must be an integer p in Zk with**

$$\begin{gathered} p(p-1) \equiv 0(\mathrm{mod}\, k) \\ 1-p \equiv p(\mathrm{mod}\, k), or\ 2p \equiv 1 \end{gathered}$$

Therefore (1) 2p-1 = rk ,

(2) p^2 - p = sk, for some integers r and s .

From the first equation we see that 2 cannot divide k, and p cannot divide. k. Hence (p,k) = 1. But this means that, from the second equation, k must divide p-1 . So k cannot divide p + (p-1) = 2p-1 , contradicting the first equation. Q.E.D.

Problem 30:

(i) The number of geodesics on a cylinder passing through two points on the same horizontal generator is infinite. Each of them is a helix on the surface in 3-space. When the cylinder is flattened out, these become a collection of straight line segments with slope L/n, where n is the number of loops performed by the helix. [1]. The distance traveled along each segment is the

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[1] *We've all seen those performing helices at carnivals!*

hypotenuse of the triangle formed by a right triangle with vertical length 1 (that of the circumference) and horizontal length L/n.

$$d_n = \sqrt{1+\frac{L^2}{n^2}} = \frac{1}{n}\sqrt{n^2+L^2}$$

The number of segments is n. Therefore the total length traversed by a light ray on a geodesic helix with n loops is

$\sqrt{n^2+L^2}$

The inverse square of this is $\frac{1}{n^2+L^2}$**. . Since the ray may wrap around the cylinder from in both clockwise and counterclockwise orientations, this should be multiplied by 2. One therefore obtains the total intensity received at O by adding up the contribution of each geodesic. The result is:**

$$I(O) = \sum_{n=1}^{\infty} \frac{2\lambda I_0}{n^2+L^2}$$

, where λ is a conversion constant depending on the units chosen .

(ii) Rewrite the above formula as:

$$I(O) = \frac{2\lambda I_0}{L} \sum_{n=1}^{\infty} \frac{L}{n^2+L^2}$$

The assertion now follows from the lemma:

Lemma:

$$\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{L}{n^2+L^2}=\frac{\pi}{2}\ (!)$$

Proof:

Consider the integral:

$$J=\int_0^{\infty}\frac{Ldx}{x^2+L^2}=\tan^{-1}(\frac{x}{L})\,|_0^{\infty}=\frac{\pi}{2}$$

That this is equal to the infinite sum above can be seen from the chain of inequalities:

$$\frac{L}{(n-1)^2+L^2}>\int_{n-1}^{n}\frac{Ldx}{x^2+L^2}>\frac{L}{n^2+L^2}>\int_{n}^{n+1}\frac{Ldx}{x^2+L^2}>...$$

The partial sums are squeezed between the partial integrals that converge to the same limit at infinity.One should not consider this a cause for anxiety. Although one might expect that the total light intensity converging on the earth from everywhere in the universe would be of blinding intensity, (a 2-dimensional Ölbers paradox), yet, as the uniform distribution of stars covers a 2-dimensional universe, the Hubble Expansion saves us once again!

Problem 31:

(i) Travel time of Tristan from Earth to Chandra:

$$t_1 = a/v_1$$

Proper time of Tristan's journey to Chandra:

$$\tau_1 = \beta t_1 = \boxed{\frac{a\sqrt{1-({}^{v_1}\!/_{2})^2}}{v_1}}$$

Likewise, the time and proper time of Tristan's journey from Chandra to Lomonosov are:

$$t_2 = b/v_2\ ;$$

$$\tau_2 = \beta t_2 = \boxed{\frac{b\sqrt{1-({}^{v_2}\!/_{2})^2}}{v_2}}$$

The time and proper time for Isolde's trip to Lomonosov are given:

$$t_3 = b/v_3\ ;$$

$$\tau_3 = \beta t_3 = \boxed{\frac{d\sqrt{1-({}^{v_3}\!/_{2})^2}}{v_3}}$$

Under the conditions of the problem there is a combination of velocities such that

$$\boxed{\begin{array}{l} t_1 + t_2 = t_3; \\ \tau_1 + \tau_2 = \tau_3 \end{array}}$$

This translates into the equations:

$$\boxed{\begin{array}{l}(i)\ \dfrac{a}{v_1}+\dfrac{b}{v_2}=\dfrac{d}{v_3}=t_3\\ (ii)\ \dfrac{a\beta_1}{v_1}+\dfrac{b\beta_2}{v_2}=\dfrac{d\beta_3}{v_3}=\tau_3\end{array}}$$

Here is the basic algebra:

$$\boxed{\begin{array}{l}av_2+bv_1=t_3v_1v_2;\, v_2(t_3v_1-a)=bv_1;\\ v_2=\dfrac{bv_1}{t_3v_1-a};\\ \dfrac{\beta_2 b}{v_2}=\dfrac{\beta_3 d}{v_3}-\dfrac{\beta_1 a}{v_1}=\tau_3-\tau_1=\beta_3t_3-\beta_1t_1\end{array}}$$

By substitution one derives

$$\boxed{\begin{array}{l}av_2+bv_1=t_3v_1v_2;\, v_2(t_3v_1-a)=bv_1;\\ v_2=\dfrac{bv_1}{t_3v_1-a};\\ \dfrac{\beta_2 b}{v_2}=\dfrac{\beta_3 d}{v_3}-\dfrac{\beta_1 a}{v_1}=\tau_3-\tau_1=\beta_3t_3-\beta_1t_1\end{array}}$$

Squaring both sides does not change the equality. Squaring and substituting, the left side becomes :

$$\boxed{\begin{aligned}&(\frac{\beta_2 b}{v_2})^2 = (\tau_3 - \frac{\beta_1 a}{v_1})^2\\ &\frac{b^2(1-({}^{v_2}\!/_{c})^2)}{(v_2)^2} = \frac{b^2\left(1-\dfrac{b^2(v_1)^2}{(t_3 v_1 - a)^2}\Big/c^2\right.}{\dfrac{b^2(v_1)^2}{(t_3 v_1 - a)^2}}\\ &= \frac{(t_3 v_1 - a)^2 - \dfrac{b^2 v_1^2}{c^2}}{v_1^2}\end{aligned}}$$

The right side becomes:

$$\boxed{\tau_3^2 + \frac{a^2(1-\frac{v_1^2}{c^2})}{v_1^2} - \frac{2a\tau_3}{v_1}\beta_1}$$

Equating right and left, then multiplying through by $(v_1)^2$ gives :

$$\boxed{(t_3 v_1 - a)^2 - \frac{b^2 v_1^2}{c^2} = \tau_3^2 + a^2(1-\frac{v_1^2}{c^2}) - 2a v_1 \tau_3 \beta_1}$$

Collecting terms in $(v_1)^2$ and v_1 , and noticing that the constant a^2 cancels out on both sides, one has:

$$\boxed{\begin{aligned}&v_1^2(t_3^2 - \tau_3^2 + \frac{a^2}{c^2} - \frac{b^2}{c^2}) = 2t_3 a v_1 - 2a v_1 \tau_3 \beta_1\\ &= 2at_3(1-\beta_1\beta_3)\end{aligned}}$$

Now:

$$\boxed{t_3^2 - \tau_3^2 = \frac{d^2}{v_3^2} - \frac{d^2}{v_3^2}(1 - \frac{v_3^2}{c^2}) = \frac{d^2}{c^2} \qquad (!)}$$

Finally one gets, after dividing both sides by v_1 , and ad :

$$\boxed{\begin{aligned}&({}^{v_1}\!/_{c})\frac{(a^2 + d^2 - b^2)}{2ad} = \frac{t_3 c}{d}(1 - \beta_1\beta_3)\\ &= \frac{(1-\beta_1\beta_3)}{({}^{v_3}\!/_{c})}\end{aligned}}$$

Using the notation suggested in the statement of relativity problems, this can be written as:

$$\boxed{u_1 u_3 h_1 = (1 - \beta_1\beta_3)}$$

The Principle of Relativity states that no material object can attain the speed of light. Therefore $\boxed{0 < u_1, u_3, \beta_1, \beta_3 < 1}$**. The velocities are assumed larger than 0 because it is assumed that the space ships will not be doubling back on their own trajectories. Therefore the right hand side of the above equation must be positive, while the left-hand side depends on the sign of h_1 . The equations are symmetrical in v1 and v2 ; therefore:**

It must be the case that both h1 and h2 are **>***0* **.**

We will now show that a contradiction results if $h_1 < 1$ or $h_2 < 1$.

(i) The assumption that h_1 and h2 > 1 leads to (i)

In that case :

$$\begin{array}{l} a^2 + d^2 - b^2 \ge 2ad; \\ b^2 + d^2 - a^2 \ge 2bd; \\ d^2 - 2ad + a^2 \ge b^2; b^2 - 2bd + d^2 \ge a^2; \\ (d-a)^2 \ge b^2; (d-b)^2 \ge a^2; \\ \therefore d \ge a + b \end{array}$$

The conclusion results from taking the square root on both sides of each inequality and examining the various possibilities. If d= a+b , the situation is equivalent in every respect to that in which the 3 planets are on the same straight line, or in which d is a full semi-circle.

(ii) Now assume that $h_1 < 1$. The calculations are as follows:

$$\begin{array}{l} u_1 u_3 h = 1 - \beta_1 \beta_3; \\ \sqrt{1-u_1^2}\sqrt{1-u_3^2} = 1 - u_1 u_3 h; \\ (1-u_1^2)(1-u_3^2) = 1 - u_1^2 - u_3^2 + u_1^2 u_3^2 = 1 - 2u_1 u_3 h + u_1^2 u_3^2 h^2; \\ u_1^2(1 + u_3^2 h^2 - u_3^2) - 2u_1 u_3 h + u_3^2 = 0 \end{array}$$

This is a quadratic in the free variable u_1 ; the discriminant is:

$$\begin{array}{l} \Delta^2 = 4u_3^2 h^2 - 4u_3^2(1 + u_3^2 h^2 - u_3^2) \\ = 4u_3^2(h^2 - 1 - u_3^2 h^2 + u_3^2) \\ = 4u_3^2(h^2 - 1)(1 - u_3^2) < 0 \end{array}$$

The final expression is less than 0 because h is less than 1, and everything else is positive. This gives a complex solution for the velocity u_1 , and is thus not possible in the real world.

(iii), (iv) Since we are assuming that both h_1 and h_2 are larger than one, it will greatly simplify our calculations to write:

$$\boxed{\begin{aligned} h_1 &= \cosh\varpi \\ h_2 &= \cosh\chi \end{aligned}}$$

Given the large number of symmetries in the problem, we will solve the equations for (v_1, h_1) , then apply this solution to compute (v_2, h_2). We proceed:

$$\boxed{\begin{aligned} & u_1 u_3 \cosh\varpi = 1 - \beta_1\beta_3; \\ & 1 - u_1 u_3 \cosh\varpi = \beta_1\beta_3 = \sqrt{1-u_1^2}\sqrt{1-u_3^2}; \\ & (1-u_1^2)(1-u_3^2) = 1 + \cosh^2\varpi u_1^2 u_3^2 - 2u_1 u_3 \cosh\varpi \\ & = 1 - u_1^2 - u_3^2 + u_1^2 u_3^2 \end{aligned}}$$

Collecting terms, with u_1 as independent variable :

$$\boxed{u_1^2(\cosh^2\varpi u_3^2 + 1 - u_3^2) - 2u_1 u_3 \cosh\varpi + u_3^2 = 0}$$

For there to be a solution, the discriminant must be real, and u_1 must be less than 1. The discriminant is given by:

$$\boxed{\begin{aligned} \Delta^2 &= 4\cosh^2\varpi u_3^2 - 4u_3^2(1 + u_3^2 \sinh^2\varpi) \\ &= 4\cosh^2\varpi u_3^2 - 4u_3^2 - 4u_3^4 \sinh^2\varpi \\ &= 4u_3^2(1-u_3^2)\sinh^2\varpi \end{aligned}}$$

Here we have made use of the hyperbolic identity:

$$\boxed{\cosh^2\varpi - \sinh^2\varpi = 1}$$

Since $u_3 < 1$, the discriminant is positive, and u_1 will be a real number. By the quadratic formula:

$$\boxed{u_1 = \frac{2\cosh \varpi u_3 \pm 2u_3\beta_3 \sinh \varpi}{2(1+u_3^2 \sinh^2 \varpi)}}$$

Fortunately(!)

$$\boxed{1+u_3^2 \sinh^2 \varpi = (\cosh \varpi + \beta_3 \sinh \varpi)(\cosh \varpi - \beta_3 \sinh \varpi)}$$

There are therefore 2 solutions for u_1 :

$$\boxed{\begin{aligned} &(a)\, u_1^a = \frac{u_3}{\cosh \varpi + \beta_3 \sinh \varpi}; \\ &(b)\, u_1^b = \frac{u_3}{\cosh \varpi - \beta_3 \sinh \varpi} \end{aligned}}$$

The symmetries of the basic equations now allow us to write:

$$\boxed{\begin{aligned} &u_1^a = \frac{u_3}{\cosh \varpi + \beta_3 \sinh \varpi} \leftrightarrow u_2^a = \frac{u_3}{\cosh \chi - \beta_3 \sinh \chi} \\ &(b)\, u_1^b = \frac{u_3}{\cosh \varpi - \beta_3 \sinh \varpi} \leftrightarrow u_2^a = \frac{u_3}{\cosh \chi + \beta_3 \sinh \chi} \end{aligned}}$$

Since the hyperbolic cosine is always larger or equal to 1, we need only verify that $\boxed{u_1^b = \frac{u_3}{\cosh \varpi - \beta_3 \sinh \varpi}}$ **is less than 1. The rest follows by symmetry.**

In fact $\boxed{\frac{u_3}{\cosh \varpi - \beta_3 \sinh \varpi} < 1}$ **implies**

$$\boxed{\begin{aligned}&u_3 < \cosh\varpi - \beta_3 \sinh\varpi;\\&0 < \beta_3 \sinh\varpi < \cosh\varpi - u_3;\\&(1-u_3^2)\sinh^2\varpi = (1-u_3^2)(\cosh^2\varpi - 1) < \cosh^2\varpi - 2\cosh\varpi u_3 + u_3^2;\\&u_3^2\cosh^2\varpi - 2u_3\cosh\varpi + 1 > 0;\\&\therefore (u_3\cosh\varpi - 1)^2 > 0\ (!)\end{aligned}}$$

(v) This is always true u*nless:*

$$\boxed{u_3 = \frac{v_3}{c} = \frac{1}{\cosh\varpi} = \frac{2ad}{d^2 + a^2 - b^2}}$$

(v) When v_3 is at this critical velocity, then v_1 must be the speed of light, which is prohibited. Since the same argument applies for the velocity v_2 , it follows that under these conditions, the problem can always be solved unless v_3 is equal to one or both of the two prohibited velocities

$$,\boxed{\frac{2adc}{d^2 + a^2 - b^2}; \frac{2bdc}{d^2 + b^2 - a^2}}$$

(vi) ***The topological argument*** **: Since by assumption d < a+b , one can imagine the circular as a piece of string which can be readjusted so that Isolde's trajectory is a straight line. If a<b but d+a >b, then the upper trajectory can be straightened out into a triangle; otherwise it will be curved in some way.** ***All that really matters is that, in real Euclidean geometry, d can be deformed into a straight line*****. Since the proper time, based on the tangential velocity, doesn't depend on the shape of a**

(smooth) trajectory, one can take Isolde's trajectory ***as one's fixed reference frame*** **. Special Relativity allows one to recast the entire problem from the viewpoint of a stationary Isolde. One then has an ordinary "Twin's Paradox" , and the crew of the Tristan ages less than that of the Isolde.**

Why can't we use this argument when d > a+b ? ***Because of the triangle inequality, real Euclidean geometry makes it impossible to pull the trajectory of Isolde into a straight line, which must be the shortest distance between the two locations of Earth and Lomonosov*** **. Therefore one cannot treat the trajectory of the Isolde as a fixed reference frame.**

Problem 32:

(i) An infinite series of the form (A,x) is monotonically increasing as a function of x. Since (A,0) = 0, there can be only one positive x such that, for a given sequence of 0's and 1's, (A,x) = 1 . Since all a_i are either 0 or 1, x must lie between 1/2 and 1.

(ii) For x = 1, $a_1 = 1$, and all the rest are 0.
For x = 1/2 , $a_i = 1$ for all 1 .

Let 1/2 < x < 1 . If there is a j such that

$$\sum_{i=1}^{j} x^i = 1$$

we are finished. Otherwise, locate the first j such that

$$\sum_{i=1}^{j} x^i < 1 , \sum_{i=1}^{j+1} x^i > 1.$$

This exists, because

$$\sum_{i=1}^{\infty} x^i = \frac{x}{1-x} > 1$$

The representation we are constructing begins

$a_1, a_2, \ldots, a_j = 1; a_{j+1} = 0$. **Let** $a_{j+1} \ldots a_m = 0$

where m is the first index after j such that

$(\sum_{i=1}^{j} x^i) + x^{m+1} < 1$**.Since** $\sum_{i=1}^{\infty} x^i = \frac{1}{1-x} > 1$

in this range, a sequence of 0's will always be followed by a sequence of 1's, which will again be followed by a sequence of 0's. In this way, a set A is constructed, with its convergence to 1 virtually guaranteed by the nature of the construction.

Problem 33:

(i) $\alpha = \frac{\sqrt{5}-1}{2}$ **is the positive root of the equation**

$$y^2 + y - 1 = 0$$

If $\alpha < x < 1$ one proceeds as follows: let A(x) be the sequence constructed via the algorithm used in (ii) (Call this the "standard representation") .

Since $\frac{1}{2} < \alpha < x$ it follows that $a_1=1$, and

$$\sum_{i=2}^{\infty} a_i x^i = 1 - x; \; a_2 x^2 + \sum_{i=3}^{\infty} a_i x^i = 1 - x$$

$$a_2 + \frac{1}{x^2} \sum_{i=3}^{\infty} a_i x^i = \frac{1-x}{x^2}$$

Since $x>\alpha$, $x^2 + x > 1, \; and \; \frac{1-x}{x^2} < 1$. This implies that, (in the standard representation, $a_2 = 0$).

Next compute the coefficients $\{b_i\}$ of the standard representation of the series $\sum_{i=2}^{\infty} b_i x^i = 1$. starting with x^2 .

We can do this because x has been so chosen that

$$\sum_{i=2}^{\infty} x^i > 1$$

One can now exhibit distinct solutions (A,x)=1, and (B,x) = 1:

$$A = (1, 0, a_3, a_4, \ldots\ldots)$$
$$B = (0, 1, b_3, b_4, \ldots\ldots)$$

In particular, if $x = \alpha$, these solutions are:

$$A = (1, 0, 0, 1, 1, 1, 1\ldots\ldots\ldots)$$
$$B = (0, 1, 1, 1, 1\ldots)$$

Problem 34:

Suppose now that $\frac{1}{2} < x < \alpha$, and consider those values x which are roots of polynomials of the form

$$P(x) = \sum_{i=1}^{j} a_i x^i = 1, a_i = 0 \; or \; 1$$

Obviously $a_j = 1$. Since

$$a_1 x^{j+1} + \dots a_j x^{2j} = x^j (a_1 x + \dots a_j x^j) = x^j$$

we can *replace* the term x^j by the above expression. This gives us a new unitary representation in addition to the standard one .

To show that there is no *unique* unitary representation for any value x other than 1/2 and 1 in this range:

Let A be any infinite sequence of 1's and 0's . We can map A into a point of the Cantor Set J on [0,1/2] via the formula :

$$y = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

Note that since the elements of J consist of ternary decimals without any 2's in their entries, there is no ambiguity in the association of each A with a real number.

Define a function x = γ(y) , whose domain is on this Cantor set, by the equation (A(y), γ(y)) = 1, where A is the sequence of 0's and 1's corresponding to the ternary decimal expression for y ε J.

Theorem I **: Let y be an element of J whose ternary decimal representation is finite. That is to say**

$$y = \sum_{i=1}^{j} \frac{a_i}{3^i}, j < +\infty$$

Then

(i) The point y is a limit point *on the right* .

(ii) T he function γ(y) is continuous *on the right* .

In other words, given an ε , there is a δ , and an element y* in J, such that y*>y, |y-y*|<δ and |x*-x|<ε , where x=γ(y) , x* = γ (y*) .

Proof: **Adding 1's to the ternary representation of y a very long distance away from the jth entry, will increase y by a tiny amount (as small as one likes), and also diminish x by a very tiny amount.**

Theorem II **: Let N be an integer sufficiently far away from k. Let**

$$y^* = \sum_{i=1}^{j} \frac{a_i}{3^i} + \sum_{i=N}^{\infty} \frac{1}{3^i} = y + \frac{1}{2\bullet 3^{N-1}}; x^* = \gamma(y^*);$$

$$1 = \sum_{i=1}^{j} a_i (x^*)^i + \sum_{i=N}^{\infty} (x^*)^i = \sum_{i=1}^{j} a_i (x^*)^i + Q(x^*),$$

$$Q(x^*) = \frac{(x^*)^N}{1 - x^*}$$

Then as y --->y* on the *Cantor Set J* , x -->x* on the *full* interval [x*,x] .

Proof : **As y moves to y*, the coefficients of the infinite series Q (defined by the difference y'-y at points y' where y <y'<y*) will move through *every one* of the sequences A before arriving at (1,1,1,1,1,1,...) . Therefore the range of**

γ between x* and x must encompass every value in the interval [x*,x] .

We can now use (iv) to prove our main result. Recall that if A is the sequence corresponding to the finite ternary $y = 0.a_1a_2...a_j \ \varepsilon \ J$, then

$1 = (A,x) = (A',x)$, where A' is the sequence of entries in:

$$y' = 0.a_1a_2...a_{j-1}0a_1a_2...a_{j-1}a_j \text{ .(base 3)}$$

Recall that $\gamma(y)$ is continuous on the right. This implies

Theorem 3 : There is an interval I_y to the right of y and an interval $I_{y'}$ to the right of y' , such that for any z ε I_y there is a z' ε $I_{y'}$ with $\gamma(z) = \gamma(z')$.

The details of the proof need not concern us here.

Since the solutions x to finite equations of the form

$$P(x) = \sum_{i=1}^{j} a_i x^i = 1, a_i = 0 \ or \ 1$$

are dense in [0,1] , the result follows

Problem 35:

(i) If $\boxed{p_n = p_m + p_l}$, then

$$\boxed{\begin{array}{l} an + b = (am + b) + (al + b); \\ a(n - m - l) = b \end{array}}$$

This implies that a is a divisor of b. Since (a,b) = 1, a must =1

(ii) If $\boxed{p_n^2 = p_m^2 + p_l^2}$, then

$$\boxed{\begin{aligned}&(an+b)^2=(am+b)^2+(al+b)^2;\\&a^2n^2+2abn+b^2=\\&a^2m^2+2abm+b^2+a^2l^2+2abl+b^2;\\&a(n^2a+2bn-m^2a-2bm-l^2a-2bl)\\&=b^2+b^2-b^2=b^2\end{aligned}}$$

This implies that a is a divisor of b^2. Since (a,b) = 1, one also has (a,b^2) = 1. Therefore a = ±1, and M = Z

(iii) Suppose $\boxed{x=y+z}$ **and** $\boxed{xp_n=yp_m+zp_l}$ **. Then**

$$\boxed{\begin{aligned}&(y+z)(an+b)=y(am+b)+z(al+b);\\&\{y+z)(an)-yam-z(al)=yb+zb-(y+z)b=0.\\&\therefore(y+z)n=ym+zl\end{aligned}}$$

and the values of a and b are irrelevant to the solution.

Problem36 :

If (A) : $\boxed{p_{n_1}^2-p_{n_1}p_{n_2}+p_{n_2}^2=p_{n_3}^2}$

one has

$$\boxed{\begin{aligned}&(an_1+b)^2+(an_2+b)^2-(an_1+b)(an_2+b)=(an_3+b)^2\\&a^2n_1^2+a^2n_2^2-a^2n_1n_2+\\&2abn_1+2abn_2-ab(n_1+n_2)+b^2+b^2-b^2\\&=a^2n_3^2+2abn_3+b^2\end{aligned}}$$

The b^2 term drops out, leaving a relation linear-homogeneous in a and b ! Collecting terms:

(E:) $$\boxed{a^2(n_1^2+n_2^2-n_1n_2-n_3^2)=ab(2n_3-(n_1+n_2))}$$

or aH = bK , where:

$$\boxed{\begin{array}{l} H=n_1^2+n_2^2-n_1n_2-n_3^2 \\ K=2n_3-(n_1+n_2) \end{array}}$$

Clearly solutions independent of a and b may be obtained by setting both H and K to 0. Then:

$$\boxed{\begin{array}{l} 2n_3=n_1+n_2 \\ n_1^2+n_2^2-n_1n_2-n_3^2=n_1^2+n_2^2-n_1n_2-(\dfrac{n_1+n_2}{2})^2=0; \\ 3(n_1^2+n_2^2)-6n_1n_2=0=3(n_1-n_2)^2 \\ \therefore n_1=n_2=n_3 \;(!) \end{array}}$$

One easily verifies that setting all the variables equal to each other provides solutions for all choices of a and b .

Problem 37:

(i) Let a=12r and b = 12s , and set H = bt , K = at in equations (E) above. The factor t will be dropped for the moment and reintroduced at the appropriate place. Then :

$$\boxed{\begin{array}{l} (i)n_1^2+n_2^2-n_1n_2-n_3^2=12s \\ (ii)2n_3-(n_1+n_2)=12r; \\ n_1=2n_3-n_2-12r \end{array}}$$

Substituting in (1):

$$
\begin{aligned}
&12s = n_1^2 + n_2^2 - n_1 n_2 - n_3^2 = \\
&(2n_3 - n_2 - 12r)^2 + n_2^2 - n_2(2n_3 - n_2 - 12r) - n_3^2 = \\
&4n_3^2 + n_2^2 + (12r)^2 \\
&-4n_2 n_3 - 48rn_3 + 24rn_2 \\
&+n_2^2 \\
&-2n_2 n_3 + n_2^2 + 12rn_2 \\
&-n_3^2
\end{aligned}
$$

Transpose, collect terms, and make n$_2$ the unknown variable:

$$3n_2^2 + (36r - 6n_3) + (3n_3^2 - 48rn_3 + 144r^2 - 12s) = 0$$

Solving for n$_2$ produces:

$$
\begin{aligned}
&(a)\; n_2 = \frac{6n_3 - 36r \pm \sqrt{144rn_3 + 144s - 432r^2}}{6} \\
&= n_3 - 6r \pm x; \\
&(b)\; x = \pm\sqrt{4rn_3 + 4s - 12r^2} \\
&(c)\; x^2 = 4rn_3 + 4s - 12r^2;
\end{aligned}
$$

Symmetry allows us to choose n$_1$ as the solution with positive x, n$_2$ as the solution with negative x . Transposing equation (3) and solving for n$_3$ gives:

$$n_3 = \frac{x^2 - 4s + 12r^2}{4r} = 3r + \frac{x^2 - 4s}{4r}$$

(ii) We now reintroduce the factor t. Rewrite the above equation as:

$$\boxed{n_3 = 3rt + \frac{x^2 - 4st}{4rt}}$$

A solution may be obtained by letting x = 2t ; then

$$\boxed{n_3 = 3rt + \frac{4t^2 - 4st}{4rt} = 3rt + \frac{t-s}{r}}$$

A special solution may therefore by obtained by letting $t_0 = s$, $x_0 = 2s$:

$$\boxed{\begin{aligned} n_3^0 &= 3rs; \\ n_1^0 &= 3rs - 6r + 2s \\ n_2^0 &= 3rs - 6r - 2s \end{aligned}}$$

Problem 38:

The general solution may be obtained by letting

$$\boxed{\begin{aligned} t_k &= s + kr; \\ x_k &= 2t_k \\ n_3^k &= 3r(s+kr) + \frac{(s+kr)-s}{r} \\ &= 3kr^2 + 3rs + k \end{aligned}}$$

The values of n_1 and n_2 are therefore:

$$\boxed{\begin{aligned} n_1^k &= n_3^k - 6r + x_k \\ &= 3kr^2 + 3rs + k - 6r + 2(s + kr) \\ &= 3kr^2 + 3rs + 2r(k-3) + 2s + k; \\ \\ n_2^k &= n_3^k - 6r - x_k \\ &= 3kr^2 + 3rs + k - 6r - 2(s + kr) \\ &= 3kr^2 + 3rs - 2r(k+3) - 2s + k \end{aligned}}$$

Note that, since the equations (E) are homogeneous in a and b , a solution for tr , rs is also a solution for r,s , which is also a solution for 12r=a , 12s = b . *Therefore we have shown that solutions exist for all pairs of non-zero integers .*

Problem 39:

(i) Let

$$\boxed{\begin{aligned} P(x) &= a_0 x^e + a_1 x^{e-1} + \ldots + a_e \\ Q(x) &= b_0 x^f + b_1 x^{f-1} + \ldots + b_f \end{aligned}}$$

Without loss of generality one can assume that there are no common factors to all of $\{a_i\}$ or all of $\{b_j\}$. Also, assume exponent $e \leq f$

The basic properties of the greatest common divisor, d= (a,b) are:

(1) (a,b) = (b,a)

(2) d = (a, b± ka) = (a ± lb , b) , k and l are arbitrary integers.

(3) Assuming x, a ,b positive , then x(a,b) $\geq$ (xa ,b) $\geq$ (a,b) . One easily shows from this that (xa,yb) $\leq$ xy(a,b) . The result carries over to negative values by using the absolute value , |(a,b)|.

If G(n) = (P(n) , Q(n)) , then , using absolute values of the gcd:

$$\boxed{\begin{array}{l}(a)\,|(P(x),a_0Q(x))| \geq |G(P,Q)| \\ (b)|(P(x),a_0Q(x))| = |(P(x),a_0Q(x)-b_0x^{f-e}P(x))| \\ |(P(x),b_1-b_0a_1x^{f-1}+\ldots+(b_f-b_0a_e))| \\ = |(P(x),Q'(x))|\end{array}}$$

where the degree of Q' is at least 1 less than the degree of Q. By going back and forth in this process one can reduce the polynomials P and Q to linear forms Ax+B , Kx + L , in which either B or L or both, are non-vanishing. Continuing the process one more step, one has:

$$\boxed{\begin{array}{l}|(Ax+B,A(Kx+L))| = |(Ax+B,A(Kx+L)-K(Ax+B))| \\ = |(Ax+B,\ AL-KB)| \geq |(Ax+B,Kx+L)| \geq \ldots \geq |(P,Q)|; \\ |(Ax+B,AL-KB)| \leq |AL-KB|; \\ \therefore |(P,Q)| \leq |AL-KB|\end{array}}$$

It follows that the absolute value of |(P,Q)|, and therefore G(n) = (P,Q), has only finitely many values. Clearly the lack of a common algebraic factor is crucial to this argument, since its presence brings the process to an abrupt halt with the appearance of 0 on one side or the other.

(ii) Let n be given, and suppose that (P(n),Q(n) = m
Then (P(n+km), Q(n+km)) =mh , as one can see by expanding the terms $(n+km)^j$, j = 1,2,. . in each of the polynomials . Since the range of G(n) is

finite, there must be a maximum h =H for any given m. It follows that G(n) has period mH .

Since a periodic function of finite range over the integers cannot have more than one period, we have also shown the following: Let m_1 , m_2 , ... m_k be the distinct integers in the range of G(n) . Then MaxG(n) = period G(n) = Lowest Common Multiple (m_1 , m_2 , ... m_k)

(iii)

$$\boxed{P(x) = ax^2 + bx = n}$$

a, b , n integers, (a,b) = 1, a>1 . If r=p/q is a solution, then q must be a divisor of a. For:

$$\boxed{\begin{array}{l} a(\frac{p}{q})^2 + b(\frac{p}{q}) = n; \\ ap^2 + bpq = nq^2; \\ \therefore ap^2 = q(nq - bp) \end{array}}$$

Since (p,q) = 1, q is a divisor of a. Write a = qd. Substituting in the above:

$$\boxed{\begin{array}{l} a(\frac{p}{q})^2 + b(\frac{p}{q}) = qd(\frac{p}{q})^2 + b(\frac{p}{q}) \\ = \frac{p}{q}(dp + b) = n \end{array}}$$

Since (p,q) = 1 q must divide dp + b. So let :

$$\boxed{\begin{array}{l} k = \dfrac{dp+b}{q}; n = kq; \\ p = \dfrac{kq-b}{d} \end{array}}$$

Suppose that q and d have a common factor , t > 1. Since by assumption (b,a) = 1, it follows that (b,d) =(b,q) = 1. In that case p cannot be an integer. It follows that q must be a *sharp divisor* of a, defined as follows:

We will say that q is a sharp divisor of a if a = qd , and (q,d) = 1. Our denominators therefore must be sought among the sharp divisors of a. The above equation for p then translates into a congruence: $\boxed{kq \equiv b(\mathrm{mod}\, d)}$. From elementary number theory one knows that if (q,d) = 1, then one can always find a solution k to this congruence. Let k_0 be such a solution, with $1 \leq k_0 < d$. Then there is a whole sequence of solutions $\{ k_m \}$; $k_m = k_0 + md$, m = 0, ±1,±2, ... , with corresponding solutions $\{ p_m \}$, for p It follows that

$$\boxed{\begin{aligned} p_m &= \frac{k_m q - b}{d} = \frac{(k_0 + md)q - b}{d} \\ &= \frac{k_0 q - b}{d} + m = p_0 + m \end{aligned}}$$

Once again, since (q,d) = 1, the set $\{ p_m \}$ represents all solutions for a given sharp divisor q of a. If S is the set of all sharp divisors of a $S = (q_1, q_2, \ldots\ldots q_k)$, then we can write the complete set of solutions for P(x) = n , as

$$\boxed{Q = \{p_0(q_j) + m\}, q_j \in S; m = 0, \pm 1, \pm 2, \ldots}$$

Problem 40:

Once again , $\boxed{P(x) = ax^2 + 2bx = -n}$, a prime ,b, n integers, (a,b) = 1; but now we look for solutions over the *complex* rationals, $\boxed{r = \frac{u + iv}{w}}$, u,v,w integers, g.c.d.(u,v,w) = 1. The equation for P(x) is a simple quadratic, so one may apply the quadratic formula to P(r) +n = 0 , to get:

$$\boxed{\begin{array}{l} r=-\dfrac{2b\pm\sqrt{4b^2-4an}}{2a}=-\dfrac{b\pm\sqrt{b^2-an}}{a}=\dfrac{u+iv}{w} \\ \therefore (i)\ w=a;u=-b \\ (ii)\ (b^2-an)=-v^2<0 \\ (iii)(v^2+b^2)=an \\ (iv)\ v^2\equiv -b^2 \pmod a \end{array}}$$

In this problem it is assumed that a is prime. Theorems about quadratic residues enable one to generalize to all a . It is clear that:

(a) If b and -b are both quadratic residues of a , or if neither b nor -b are quadratic residues, then -b^2 is a quadratic residue of a.

(b) If b is a quadratic residue, and -b is not a quadratic residue of a , or vice versa, then -b^2 is a quadratic non-residue of a.

Therefore, one can find a solution (c,-n) if and only if -1 is a quadratic residue of a . One now invokes a basic theorem of Number Theory:

Theorem : -1 is a quadratic residue of a, if and only if a is of the form 4n+1 . **(See for example Topics in Algebra, i.n.herstein, Wiley and Sons, 1975; pg. 360) . Therefore a must be a prime of this form.**

□□□□□□□□□
□□□□□□□

Problem 41:

$$\boxed{P(x)=a_0x^3+2a_1x^2-a_2x}$$

all coefficients are non-zero integers , a_0 >2 ; $\boxed{(a_0,2a_1)=1}$ **(so that in particular, a_0 is odd) .**

From the previous problem we know that q must divide a_0 . Requiring that a_0 and a_1 be relatively prime is enough to show, (using

arguments along the lines of those i problem 10) , that q must be a sharp divisor, or a_0 = qd, (q,d) = 1. We now proceed as follows:

$$\boxed{\begin{array}{l} a_0(\frac{p}{q})^3 + 2a_1(\frac{p}{q})^2 - a_2(\frac{p}{q}) = n; \\ a_0p^3 + 2a_1p^2q - a_2pq^2 = nq^3 \end{array}}$$

As a_0 =dq, this can be divided by q:

$$\boxed{dp^3 + 2a_1p^2 - a_2pq = nq^2}$$

Since (p,q) = 1, this equation shows that p must be a divisor of n, or n =pe .

Substituting back into the equation and dividing through by p:

$$\boxed{\begin{array}{l} dp2 + 2a_1p - a_2q = eq^2; \\ dp^2 + 2a_1p - q(eq + a_2) = 0 \end{array}}$$

This is a quadratic in the variable p. Solving:

$$\boxed{p = \frac{-a_1 \pm \sqrt{a_1^2 + d(eq^2 + a_2q)}}{d} = \frac{-a_1 + k}{d}}$$

Once again it is required that the radical be a perfect square: write :

$$\boxed{\begin{array}{l} k^2 = (a_1^2 + d(eq^2 + a_2q)); \\ \therefore e = \dfrac{k^2 - a_1^2 - da_2q}{dq^2} \end{array}}$$

From the equation for p one sees that $\boxed{k = a_1 + pd}$**. A further substitution gives:**

$$\boxed{\begin{array}{l} e = \dfrac{(a_1 + pd)^2 - a_1^2 - a_2dq}{dq^2} \\ = \dfrac{dp(dp + 2a_1) - a_2dq}{dq^2} \end{array}};$$

$$\boxed{e = \frac{p(dp + 2a_1) - a_2 q}{q^2}}$$

Since (q,p) = 1, q must divide $dp+2a_1$. Hence $sq = dp + 2a_1$, or $sq - 2a_1 = dp$. Since a_1 is given, $(a_0,2a_1)=1$, q has to be a sharp divisor of a_0

$d = a_0/q$, (q,d) =1 Therefore there exist solutions of the congruence $\boxed{sq \equiv 2a_1 (\mathrm{mod}\, d)}$. Let s_0 be a solution such that $0 < s_0 < d$. Then

$$\boxed{s_0 q - 2a_1 = dp_0;}$$

Other solutions are:

$$\boxed{\begin{aligned} s_m &= s_0 + md \\ p_m &= p_0 + mq \end{aligned}}$$

s0 and p0 having been calculated, "m" now becomes the independent variable. Substituting in the above equation , one obtains a set of solutions $\{e_m\}$:

$$\boxed{\begin{aligned} e_m &= \frac{p_m(dp_m + 2a_1) - a_2 q}{q^2} \\ &= \frac{(p_0 + mq)(r_0 + md)q - a_2 q}{q^2} \\ &= \frac{(p_0 + mq)(r_0 + md) - a_2}{q} \\ &= m(r_0 + md) + \frac{p_0 r_0 + p_0 md - a_2}{q} \end{aligned}}$$

We may therefore find the set of values for m from the congruence:

$$\boxed{(p_0 d)m \equiv a_2 - p_0 r_0 (\mathrm{mod}\, q)}$$

Since (p,q) =(q,d) = 1, there is a minimal solution m_0 such that

0 < m_0 < q. The solution set for m therefore contains numbers of the form $\boxed{m_j = m_0 + qj;\ j = 0, \pm 1, \pm 2, \ldots}$

In conclusion, for a given sharp divisor q of a0 , the set of fractional solutions {r_j } = {p_j/q} of the Diophantine equation P(rj) = n = integer, is

$$\boxed{\begin{aligned} &\frac{p_j}{q} = \frac{p_0}{q} + m_j = \frac{p_0}{q} + (m_0 + jq); j = 0, \pm 1, \pm 2; \\ &n = e_j p_j; \\ &e_j = m_j(s_0 + m_j d) + k_0 + p_0 dj \end{aligned}}$$

where $\boxed{k_0 = \dfrac{(p_0 d)m_0 + p_0 r_0 - a_2}{q}}$

Problem 42:

$$\boxed{(a + \sqrt{d})^3 + (a - \sqrt{d})^3 = b^3}$$

Multiplying out the terms on the left side one gets :

$$\boxed{2a^3 + 6ad = b^3}$$

This factors as :

$$\boxed{2a(a^2 + 3d) = b^3}$$

We see that b is even and that a divides b^3 . We will deal with the factor of 2 in a moment . Writing b = xyzw ,

$$\boxed{\begin{aligned} &b^3 = x^3 y^3 z^3 w^3 = (x^3 y^2 z)(y z^2 w^3) \\ &= a(y z^2 w^3) \end{aligned}}$$

Then

$$\boxed{\begin{aligned} &a = (x^3 y^2 z); \\ &a^3 = x^9 y^6 z^3 \end{aligned}}$$

In other words, x is the largest cube in a that is a factor of b, y is the largest square, z contains neither square nor cubic factors, and w does not enter in a at all. There is some ambiguity in this factorization: if y and z share a factor r, and x and w share a factor t, then the factorizations (xr/t , yt/r , zt/r, wr/t) and (x, y,z,w) are equivalent.

Substituting :

$$\boxed{2x^9y^6z^3+6x^3y^2zd=x^3y^3z^3w^3}$$

Dividing through by a:

$$\boxed{2x^6y^4z^2+6d=yz^2w^3}$$

Finally:

$$\boxed{\begin{array}{l}6d=yz^2(w^3-2x^6y^3);\\ a=x^3y^2z;\\ b=xyzw\end{array}}$$

Since b must be even, at least one of the numbers x,y,z, w is also even. The presence of a factor of 2 in each leads to various limitations on a,b and d :

(1) y = 2γ . Then:

$$\boxed{b=2x\gamma zw;\ a=4x^3\gamma^2z;\ 3d=\gamma z^2(w^3-16x^6\gamma^3)}$$

a has a factor of 4

---

(2) z = 2ζ . Then :

$$\boxed{b=2xy\zeta w;\ a=2x^3y^2\zeta;\ 3d=2y\zeta^2(w^3-2x^6y^3)}$$

d has a factor of 2

---

(3) w = 2ω . Then :

$$\boxed{b = 2xyz\omega; a = 2x^3y^2\varpi, 3d = yz^2(4w^3 - x^6y^3)}$$

---

(4) x = 2ξ . From the expression for d in equation (C) one sees that if x is even then one of the other variables must also be even. Taking each case separately:

(i) y = 2γ . Then:

$$\boxed{b = 4\xi\gamma zw; a = 32\xi^3\gamma^2 z; 3d = \gamma z^2(w^3 - 1024\xi^6\gamma^3)}$$

(ii) z = 2ζ . Then :

$$\boxed{b = 4\xi\zeta yw; a = 16\xi^3y^2\zeta; 3d = 2y\zeta^2(w^3 - 128\xi^6\gamma^3)}$$

(iii) w = 2ω . Then :

$$\boxed{b = 4\xi\omega yz; a = 8\xi^3y^2z; 3d = 4yz^2(\omega^3 - 16\xi^6y^3)}$$

From the equation for d in (C) one sees that either

(i) 3 divides y

(ii) 3 divides z , or:

(iii) $\boxed{w^3 \equiv 2x^6y^3 (\mathrm{mod}\, 3)}$**. Since** $\boxed{\forall k(k^3 \equiv k(\mathrm{mod}\, 3))}$

one has $\boxed{w \equiv 2x^2y \equiv -x^2y(\mathrm{mod}\, 3)}$

If x is divisible by 3, then so is w. If not, then $\boxed{w \equiv -y(\mathrm{mod}\, 3)}$

Problem 43:

$$\boxed{(a + 6i\sqrt{2})^3 + (a - 6i\sqrt{2})^3 = b^3}$$

Proceeding as before, one obtains the equation:

$$\boxed{2a^3 - 432a = b^3 = 2a(a^2 - 6^3)}$$

d = -72 . Substituting this into equation (C) and experimenting with the values of x,y,z, w, one obtains two solution sets:

$$\boxed{\begin{array}{l} I: x=1,\ y=4,\ z=6,\ w=5; \\ \quad a_1=\pm 96;\ b_1=\pm 120 \\ \\ II: x=1,\ y=2,\ z=3,\ w=-2; \\ \quad a_2=\pm 12;\ b_1=\mp 12 \end{array}}$$

Problem 44:

(i) Our first step is to show that both a and b are divisible by 4. Clearly b is even, b = 2b' . Hence:

$$\boxed{\begin{array}{l} 2(a^3-6^3a)=8b'^3; \\ a^3-6^3a=4b'^3 \end{array}}$$

This equation shows that a is even , a = 2a' . Hence:

$$\boxed{\begin{array}{l} 8a'^3-6^3\bullet 2a'=4b'^3; \\ 2a'^3-4\bullet 27a'=b'^3 \end{array}}$$

which shows once again that b' is even b' = 2β. Substituting:

$$\boxed{\begin{array}{l} 2a'^3-4\bullet 27a'=8\beta^3; \\ a'^3-54a'=4\beta^3 \end{array}}$$

This shows that a' is even , a' =2α :

$$\boxed{\begin{array}{l} 8\alpha^3-108\alpha=4\beta^3; \\ 2\alpha^3-27\alpha=\beta^3 \end{array}}$$

Thus , a and b are divisible by 4

Our problem has been reduced to finding the solutions:

$$\text{(D)}: \boxed{2\alpha^3-27\alpha=\beta^3}$$

As a way of checking the work so far, one can verify to one's satisfaction that α = 24 ; β = 30 , and α = 3 , β = -3 , are solutions.

Put equation (D) in the form:

$$\boxed{\alpha = \frac{2\alpha^3 - \beta^3}{27}}$$

We will obtain a contradiction by assuming that neither α nor β contain a factor of 3 .(If one does then clearly the other must also). There are two possibilities:

(1) α and β are of the form: $\boxed{\alpha = 3h+2; \beta = 3j+1}$

(2) u and v are of the form: $\boxed{\alpha = 3h+1; \beta = 3j+2}$

In the first case one multiplies through to get:

$$\boxed{\begin{aligned} 3h+2 &= \frac{(54h^3+108h^2+72h+16)-(27j^3+27j^2+9j+1)}{27} \\ &= (2h^3+4h^2-j^3-j^2)+\frac{5-3j+24h}{9} \end{aligned}}$$

Since 3 does not divide 5, this cannot be an integer. In the second case one gets:

$$\boxed{3h+1 = \ldots = (2h^3+2h^2-j^3-2j^2)+\frac{12j-6h-2}{9}}$$

Since 3 doesn't divide 2 this can't be an integer.

It follows that α and β must both be divisible by 3, and therefore that a and b are divisible by 12.

(ii) Dividing a and b through by 12 one ends up with an equation of the form:

$$\boxed{2u^3 - 3u = v^3}$$

Let v =xyzw , u = x^3y^2z. Then $\boxed{D = yz^2w^3 = 3^e Q}$

Once more working through the details the auxiliary equation becomes:

(E) : $\boxed{3 = yz^2(2x^6y^3 - w^3)}$

(i) e = 0 implies y = z = 1. The two solutions presented in Problem 14 satisfy this condition.

e = 2 . Then y = 9 is clearly too large . So y = 1, z= 3 is the only possibility. But then z^2 = 9, which is too large for the right side.

(ii) e = 3 is only compatible with w = 3, y = z = 1. But if w = 3, then x must also = 3, which gives a 27 on the right hand side. Obviously for any value e >3, y,z and w must all contain a factor of 3, making the right side too large.

Problem 45:

The situation for e = 1 touches on advanced topics in number theory, and the author frankly doesn't know what the answer is. Here is a sample of what one is up against:

If e = 1, then y = 3. Equation (E) becomes:

(F): $\boxed{54x^6 - w^3 = 1}$

This can be manipulated to produce:

$$\boxed{\begin{array}{l} 54x^6 = w^3 + 1; \\ w^3 + 1 = (w+1)(w^2 - w + 1) \\ w^2 - w + 1 = (w+1)^2 - 3w \end{array}}$$

From the final equation one sees that the greatest common divisor of the two factors of w^3 + 1 can only be 1 or 3, and that if w+1 is only divisible by 3, then w^3+1 can be at most divisible by 9. Since there is a 27 on the left, w+1 must be divisible by 9 . Furthermore, w+1 must be even. It follows that one can decompose x into two relatively prime factors p and q, such that :

$$\boxed{\begin{array}{l} x = pq \\ w+1 = 18p^6 \\ w^2 - w + 1 = 3q^6 \end{array}}$$

Substituting from the first upper equation to the lower yields:

$$\boxed{\begin{array}{l} (18p^6)^2 - 3(18p^6 - 1) = 3q^6 \\ = 2^2 3^4 (p^6)^2 - 2.3^3 p^6 + 3; \\ 2^2 3^3 (p^6)^2 - 2.3^2 p^6 + (1 - q^6) = 0 \end{array}}$$

This is an ordinary quadratic equation in the variable p^6 . Solving gives:

$$\boxed{\begin{array}{l} p^6 = \dfrac{18 + \sqrt{(2.3^2)^2 - 4.2^2 3^3 (1 - q^6)}}{2^3 3^3} \\ = \dfrac{3 + \sqrt{9 - 12(1 - q^6)}}{36} = \dfrac{3 + \sqrt{12q^6 - 3}}{36} \end{array}}$$

This equation can ultimately be reduced to the form:

$$\boxed{p^6 = \dfrac{2 + \sqrt{\dfrac{4q^6 - 1}{3}}}{24}}$$

Write :

$$\boxed{\begin{array}{l} c = 2q^3; \\ \dfrac{c^2 - 1}{3} = r^2; \\ c^2 - 3r^2 = 1 \end{array}}$$

This is a Pell's equation, which has infinitely many solutions. It is not known to the author if there are infinitely many solutions when c is restricted to the values determined by q; nor if there are, whether one can

then use those values to produce a number to the 6th power to determine the value of p. The author doesn't know the answer. Is there a number theorist in the house?

Problem 46:

If z = 1, the equation $g = n = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ **becomes**

$$\frac{x^2 + y}{xy} = n - y = m$$

The obvious solutions are x = 1, y = 1 , m=2, n = 3

x=2 , y = 4, n= 5

We will show that these are the only solutions for x,y >0 , z = 1 .

From the above equation one sees that x must divide y . So set y = kx . Substituting:

$$\frac{x^2 + kx}{kx^2} = \frac{x + k}{kx}$$

This shows that x must divide k , or k = hx . Once again:

$$\frac{x + hx}{hx^2} = \frac{1 + h}{hx}$$

Since h must divide 1, h = 1, and x can only equal 1 or 2 . Since y =kx = hx^2 , y equals 1 or 4 respectively .

Problem 47

As stated z = 1 , q > 1 . Then

$$\frac{p}{q} = \frac{x}{y} + y + \frac{1}{x} = y + \frac{x^2 + xy}{xy}$$

x and y can have a common divisor . Let t = gcd(x,y); x = at, y = bt, gcd (a,b) =1 Substituting:

$$\frac{p}{q} = tb + \frac{t^2a^2 + tb}{t^2ab}$$
$$= tb + \frac{ta^2 + b}{tab} = tb + h$$

By hypothesis gcd(a,b) = 1. If t and a have a common factor h will still be in lowest terms, unless t and b have a common factor. Therefore, we let b = ls , and t = ms , with (l, m) = 1. Then :

$$h = \frac{msa^2 + ls}{lmas^2} = \frac{1}{lma}(\frac{ma^2 + l}{s})$$

Since (l.m)=1 , (a,b)=1 it follows that (l,a) = 1 , (a,s) = 1. Therefore:

$$\gcd(lma, ma^2 + l) = 1$$

Since s must divide ma1 + l, ***it follows that s is restricted by the values of l, m and a*** **.**

If $\gcd(s, ma^2 + l) = 1$**, h is already in terms, with irreducible denominator q. Otherwise these two terms share a common factor, d :**

$$\gcd(s, ma^2 + l) = d$$

We now reason as follows: Let q be given, q $\geq$2 .

Factor q into 4 divisors q = lmaf , where

(l,m) = 1 , (l,a)=1 , (a,f) = 1. q being finite, such a factorization can only be done in a finite number of ways. (Note that we do not require that

(l,f) = 1 , or (m,f) = 1)

Let W = $ma^2 + l$. Let $D = d_1, d_2, \ldots\ldots, d_j$ be the collection of divisors of W. Every denominator , N, which reduces to q must be of the form

N = lmas = lmafd, where d is a member of D. Since there are only finitely many divisors of W, and W can only be constructed from q in a finite number of ways, the total number of numerators, p, must be finite. In fact we have:

$$p_j = \frac{W}{s} + tb = \frac{W}{s} + mls^2$$

The result is true even if one of the numbers x or y is negative. If, for example, we let l be negative, then W = ma^2 - l' , l' = -l > 1 This will lead to an exceptional case only if W = 0, that is $ma^2 = l$. Since (l,m) = (l,a) = 1, this is only possible if a = 1, m = 1, l = -1. The theorem continues to be true if q = 1 , x and y > 0 , z = 1.

Problem 48:

The situation described above corresponds to the cases

n = $-m^2$, x = m, z=1, y= - m^2. Therefore there is at least one solution for every negative square n .

Problem 49:

Writing out the equation g(x,y,z) = n as a polynomial:

$$x^2 z + y^2 x + z^2 y = nxyz$$

Rearranging: $x(nyz - xz - y^2) = z^2 y$. One sees that x divides z^2y
By symmetry y divides x^2z and z divides y^2x,

Thinking about the situation one realizes that x, y and z factor in the following manner:

$$x = a_1 a_2 c_1^2 c_2;$$
$$y = a_1^2 a_2 b_1 b_2;$$
$$z = b_1^2 b_2 c_1 c_2$$

where the c's are factors shared by x and z, etc. There are no other factors since we are assuming gcd (x,y,z) = 1.

Substituting in the equation for g:

$$g = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{x^2 z + y^2 x + z^2 y}{xyz}$$

$$= \frac{a_1^2 a_2^2 b_1^2 b_2^1 c_1^5 c_2^3 + a_1^5 a_2^3 b_1^2 b_2^2 c_1^2 c_2^1 + a_1^2 a_2^1 b_1^5 b_2^3 c_1^2 c_2^2}{(a_1 a_2 b_1 b_2 c_1 c_2)^3}$$

$$= \frac{a_1^1 c_1^3 c_2^2 + a_1^3 a_2^2 b_2^1 + b_1^3 b_2^2 c_2^1}{(a_1 b_1 c_1)(a_2 b_2 c_2)^2}$$

Since gcd(x,y,z) = 1, c_2 , which is common to x and z , cannot divide y, therefore cannot divide $a_1^3 a_2^2 b_2$ **. Therefore, in fact** $a_2 = b_2 = c_2 = 1$

.

Replacing a_1 by a , b_1 by b, c_1 by c, one derives, finally:

$$g = \frac{a^3 + b^3 + c^3}{abc} = n$$ **as required.**

Problem 50

The follows solutions may be found with a bit of experimentation. The expression (a,b,c) = (m_1 , m_2 , m_3) means that the values m_1 , m_2 , m_3 may be distributed between a,b and c in any permutation:

$$(1)\ (a,b,c)=(1,1,1)$$
$$x=1, y=1, z=1, \quad n=3$$

$$(2)\ (a,b,c)=(1,1,2)$$
$$x=1, y=2, z=4 \quad n=5$$

$$(3)\ (a,b,c)=(1,2,9)$$
$$(i) x=4, y=9, z=162, \quad n=41$$
$$(ii) x=2, y=36, z=81, \quad n=41$$

$$(4)\ (a,b,c)=(1,2,3)$$
$$(i) x=9, y=2, z=12, \quad n=6$$
$$(ii) x=3, y=18, z=4, \quad n=6$$

$$(5)\ (a,b,c)=(1,5,9)$$
$$(i) x=225, y=18, z=5, \quad n=19$$
$$(ii) x=405, y=25, z=9 \quad n=19$$

Comment by Noam Elkies, Harvard University :

" E(n) : $a^3+b^3+c^3=nabc$ **is an Elliptic Curve with at least one rational point (1:-1:0) . If there is any non-torsion point then there are infinitely many; and if any component of the real locus contains such a point, then the rational points are dense in that real locus -- including the points with a,b,c all positive. That said, I would expect that some E(n) have points of infinite order, some would not, and it would be a very hard problem to predict for each n which is the case for g(n).**

"If there's a non-trivial solution in polynomials a(n), b(n), c(n) , then it yields a non-torsion point for all but finitely many n, but I doubt that this happens for an equation as simple as E(n) "

Comment by Edray Goins, Cal Tech :

" I thought some about the problem you posed, and I wanted to share some thoughts.

Conjecture: Let n be a positive integer. Then the equation x/y + y/z + z/x = n has only finitely many positive integral solutions x,y,z (up to scalar multiple).

"I want to argue that the conjecture is not true by discussing in detail the case n = 6. The substitution you showed me on Friday involving the curve $a^3 + b^3 + c^3 = nabc$ looked similar to an elliptic curve, so I thought to translate the conjecture into one explicitly involving an elliptic curve. Fix an integral solution (x,y,z) and make the substitution:

$$u = \frac{3(n^2 z - 12x)}{z}$$
$$v = 108\left(\frac{2xy - nz + z^2}{z^2}\right)$$

Then (u,v) is a rational point on the elliptic curve:

$$E_n : v^2 = u^3 + Au + B;$$
$$A = 27n(24 - n^3)$$
$$B = 54(216 - 36n^3 + n^6)$$

(It actually turns out that E(n) is an Elliptic Curve whenever n is different from 3, but I'll discuss that case separately.) Now this curve has the "obvious" rational point T = ($3n^2$, 108), which has order 3, considering the group structure of E_n . It actually turns out that these 3 multiples correspond to the cases x = 0 and z = 0, so if such an integral solution (x,y,z) exists then the rational solutions (u,v) must correspond to a point on E_n not of order 3.

"Now I decided to explicitly compute points on E(n) for various values of n to see what would happen. In the following table I'm computing the Mordell-Weil group of the rational points on the Elliptic Curve, i.e. the group structure of the set of rational solutions (u,v) :

n =1 ; $E_n(Q) = Z_3$

n =2 ; $En(Q) = Z_3$

n =3 ; ...see below...

n =4 ; $En(Q) = Z_3$

n =5 ; $En(Q) = Z_6$

n =6 ; $En(Q) = Z_3 x Z$

n =7 ; $En(Q) = Z_3$

n =8 ; $En(Q) = Z_3$

n =9 ; $En(Q) = Z_3 x Z$

"Hence when n =1 ,2,4,7 or 8 we find no integral solutions (x,y,z) . When n=5 there are only 6 rational points on E_n , namely the multiples of (u,v) = (3,756) , which all yield just one positive integral point (x,y,z) = (2,4,1) ... But something fascinating happens when n = 6 ...

"The rank in all the previous cases is 0, so En has only finitely many points, thereby proving the conjecture in these cases. However when n = 6 the rank is positive (the rank is actually 1) so there are infinitely many rational points (u,v) . But we must be careful: not all rational points (u,v) yield positive integral points (x,y,z) . Clearly we can scale z large enough to always choose x and y to be integral, but we might not have both x and y positive. You'll note that x >0 if and only if u < $3n^2$, so we only want rational points in a certain region of the graph. Since the rank is 1, this part of the graph is dense with rational points!

Hence, if we can choose n so that E_n has positive rank , then I would expect the above conjecture
(that En has at most finitely many solutions for all n) to be false.

"Let me give some explicit numbers. When n = 6 , the torsion part of $E_n(Q)$ is generated by T = (108 ,108) , and the free part is generated by (u,v) = (-108,2052) . By considering various multiples we get a lot of positive integral solutions - yet unwieldy! - points (x,y,z) such that

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=6:$$

(1) 12 ; 9 ; 2

(2) 17415354475 ; 90655886250 ; 19286662788

(3) 260786531732120217365431085802 ;1768882504220886840084123089612

; 1111094560658606608142550260961

(4) 64559574486549980317349907710368345747664977687333438285188 ;

70633079277185536037357392627802552360212921466330995726803 ;

31381830303893596780062940130789557072745299086647462868546 .

"I'll just mention in passing that when n = 9 the Elliptic Curve E_9 also has rank 1. The generator (u,v) =(54,4266) corresponds to the positive integral point (x,y,z) = (63,98,12) on this curve.

What about n = 3? The curve E_n becomes $v^2=(u-18)(u+9)^2$

This gives two possibilities, either u= - 9 , or u = 18. The first corresponds to x = z, while the second corresponds to z/x > 4 . By cyclically permuting x, y and z, we find similarly that either x =y =z , or

$x/y + y/z + z/x \geq 6$. The latter case cannot happen by assumption, so $x = y = z$ is the only possibility, i.e. $(x,y,z) = (1,1,1)$ is the only solution."

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