Geodesics on the 3-Cone

A Supplement to "Conical Gravity"

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January 25,2005

Let us entertain the possibility that the physical universe we see around us can be modeled by the 3-dimensional surface of a cone in 4 dimensions. For the moment one can assume that this universe is devoid of relativistic phenomenon, but that the self-gravity along self-intersecting geodesics which has been described in the previous sections may occur. How would this be interpreted by an observer at rest, say on the planet Earth?

It has not yet been established that geodesics on the 3-D surface of a cone in R4 will in fact $\boxed{\alpha_1 = \frac{k_1}{D^2}}$ be self-intersecting, however if they are there will be an effective acceleration of all objects, galaxies for example, in the universe towards the vertex at some distant location. Since, as we have seen, this acceleration is inversely proportional to the square of the distance from the vertex, each galaxy will appear to be accelerating from every other galaxy. Let us suppose that the distance to the vertex, D, is enormous, and that the distance between two galaxies is some number of light years, a. Then, suppose that the attraction of Galaxy I to the vertex is $\boxed{\alpha_1 = \frac{k_1}{D^2}}$, and that the attraction of Galaxy II is given by $\boxed{\alpha_2 = \frac{k_2}{(D+a)^2}}$. Then the acceleration of

Galaxy II with respect to Galaxy I is

$$\boxed{\begin{aligned}\alpha &= \frac{k_1}{(D)^2} - \frac{k_2}{(D+a)^2}\\ &= \frac{D^2(k_1-k_2)+2k_1Da+k_1a^2}{(D)^2(D+a)^2}\end{aligned}}$$

If k1 and k2 are very close, the leftmost term can be neglected, and if a is very small compared to D, we can neglect the right most term. This leaves the middle term, that is to say:

$$\boxed{\alpha \approx \frac{2k_1Da}{(D)^2(D+a)^2} \approx \frac{2k_1a}{D^3}}$$

The requirements now become very special, but by juggling k_1, k_2 , a and D, one can argue that the acceleration is linear in a. Also, by modifying the attractive force in some fashion, one could possibly model the Hubble Field by the self-attraction of galaxies on a 3-D conical surface under ordinary Newtonian gravitation. It's something worth looking into, and it may give some insight into the nature of dark matter and dark energy.

We are working in R^4 with coordinates (x,y,z,w) . In analogy with the 3-dimensional case, the equation of a cone is given by

$$\boxed{E: w^2 = k^2(x^2+y^2+z^2)}$$

(1) What do geodesics on this surface, other than the generator lines, look like?

(2) Are they similar?

(3) Are they self-intersecting?

(4) How does one compute the self-gravity?

(5) Can one, in analogy to the 3-dimension case, "unfold" the surface, flattening it out and studying the local structure as a portion of 3-dimensional Euclidean space?

Some of these questions can be answered immediately. Any smooth ruled surface (one formed from a sub-pencil of lines emanating from the origin), has a locally Euclidean structure because the curvature is 0 at every point.

Even as a straight line in 3-space can be described as the intersection of non-parallel planes, so geodesics on the 3D cone surface, (which we now designate as K^3) , can be described as the intersection of surfaces which, relative to the cone, are locally Euclidean planes.

The obvious way of describing the "unfolding" or "flattening" of K^3 , is to express the intrinsic metric on K^3 in terms of both its intrinsic spherical coordinates and in terms of 4-D spherical coordinates in R^4, then setting them equal to each other.

4-Dimensional Spherical Coordinates

These combine the radius vector, r, from the origin, with the direction cosines α, β, γ of Cartesian coordinates x,y and z. It is convenient when working with the cone to put these into the following form:

$$\boxed{\begin{aligned} w &= r\cos\alpha \\ x &= r\sin\alpha\cos\beta \\ y &= r\sin\alpha\sin\beta\cos\gamma \\ z &= r\sin\alpha\sin\beta\sin\gamma \end{aligned}}$$

As the generator lines of K3 are the radius vectors from the origin, its equation in terms of these coordinates is simply:

$$\boxed{\alpha = \tan^{-1}(k) = const.}!$$

In computing the metric on K³ in terms of these coordinates, the fact that dα = 0 will be very useful.

$$\boxed{\begin{aligned}&dw = dr\cos\alpha\\&dx = dr\sin\alpha\cos\beta - r\sin\alpha\sin\beta d\beta\\&dy = dr\sin\alpha\sin\beta\cos\gamma + r\sin\alpha\cos\beta\cos\gamma d\beta\\&-r\sin\alpha\sin\beta\sin\gamma d\gamma\\&dz = dr\sin\alpha\sin\beta\sin\gamma + r\sin\alpha\cos\beta\sin\gamma d\beta\\&-r\sin\alpha\sin\beta\cos\gamma d\gamma\end{aligned}}$$

The square of the metric is

$$(d\rho)^2 = (dx)^2 + (dy)^2 + (dz)^2 + (dw)^2$$

In making the substitutions and calculations, one can quickly convince oneself that all of the "cross-terms", those in dαdβ or drdβ for example, cancel out , and one is left with:

$$\boxed{\begin{aligned}&(d\rho)^2 = (dx)^2 + (dy)^2 + (dz)^2 + (dw)^2\\&= dr^2\{\cos^2\alpha + \sin^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta\cos^2\gamma\\&+\sin^2\alpha\sin^2\beta\sin^2\gamma\}\\&+r^2\{\sin^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta\cos^2\gamma + \sin^2\alpha\sin^2\beta\sin^2\gamma\}\\&\ldots = dr^2 + r^2\sin^2\alpha[d\beta^2 + \sin^2\beta d\gamma^2]\end{aligned}}$$

This is the metric on K3 in terms of extrinsic spherical coordinates. Since α is constant, sinα is also constant, and in fact $\boxed{\sin\alpha = \frac{1}{\sqrt{1+k^2}} = const. \equiv h}$. The metric assumes the simple form:

$$\boxed{(d\rho)^2 = dr^2 + r^2h^2[d\beta^2 + \sin^2\beta d\gamma^2]}$$

The metric in terms of intrinsic spherical coordinates is simply that of Euclidean 3-space. Let the Cartesian coordinates in

K³ be designated p,q,s, the spherical coordinates by l , ψ , θ .
Then,

$$\boxed{\begin{aligned}
&p = l\cos\psi\\
&q = l\sin\psi\cos\theta\\
&s = l\sin\psi\sin\theta\\
&(d\rho)^2 = (dp)^2 + (dq)^2 + (ds)^2\\
&= (dl\cos\psi - l\sin\psi d\psi)^2 +\\
&(dl\sin\psi\cos\theta + l\cos\cos\theta d\psi - l\sin\psi\sin\theta d\theta)^2\\
&+(dl\sin\psi\sin\theta + l\cos\psi\sin\theta d\psi + l\sin\psi\cos\theta d\theta)^2\\
&\ldots = dl^2 + l^2[d\psi^2 + \sin^2\psi d\theta^2]
\end{aligned}}$$

By equating these two forms of the metric we can examine the unfolded or flattened hyper-surface of K3, actually its projection, in terms of the coordinates r,α,β, γ . This representation is analogous in every way to the treatment presented in the previous section of geodesic structure on the surface of a cone in 3-space. The projection preserves intersections, which is what we are looking for. Equating the two forms of the metric:

$$\boxed{dl^2 + l^2[d\psi^2 + \sin^2\psi d\theta^2] = (d\rho)^2 = dr^2 + r^2h^2[d\beta^2 + \sin^2\beta d\gamma^2]}$$

By equating corresponding coordinates (this is equivalent to projection) one obtains:

$$\boxed{\begin{aligned}
&(1) dl^2 = dr^2; or\ l = r\\
&(2) r^2h^2d\beta^2 = l^2d\psi^2; or\ h\beta = \psi\\
&(3) h^2\sin^2\beta d\gamma^2 = \sin^2\psi d\theta^2 = \sin^2(h\beta)d\theta^2;\\
&or\ d\theta = \frac{h\sin\beta}{\sin(h\beta)}d\gamma
\end{aligned}}$$

Clearly there are difficulties involved in the relationship between θ and γ . In order to solve this equation, one considers some equation on the hyper-surface K^3 of the form $f(l, \theta, \psi) = 0$. Solving this allows one to substitute an expression in l and ψ into equation (3), which can be translated into a differential equation in the coordinates r,β and γ . Intersection with a second curve $g(l, \theta, \gamma)$ enables one to eliminate l and express θ entirely in terms of γ .

Even in the case of a linear relationship between the coordinates p,q and s, (that of a plane in the conic hyper-surface), this procedure is extremely cumbersome. However it is not difficult to show, informally, that all geodesics on K3 which are not generator lines, are similar. By choosing a geodesic whose equation is of a particularly simple form, we will then understand the intersection structure of all geodesics.

To show why all geodesics are similar, observe that the equation for K^3 is homogeneous in w, x, y and z . By its very structure, it is also rotationally invariant in the coordinates γ and β , while a is constant . Furthermore it is locally Euclidean at each point. This means that the cone "looks the same", at all points, i.e., it is self-similar.

Now let P be a plane in the hyper-surface of K3 . One can drop a perpendicular line L from the vertex to P. It is not difficult to see that the structure P and L combined, can be rotated, then reduced or expanded by a similarity transformation to any other combination P' plus L' .

If G is any geodesic line on K^3, (not a generator), one can likewise drop a perpendicular L from the vertex O onto G, then pass a plane P through G which is perpendicular to L.

This structure (P,L,G), will then be similar to all other structures (P',L', G') on K^3. Hence all geodesics are similar.

The argument is not rigorous, although it closely follows similar arguments which show that all non-generator geodesics on the surface of a cone in 3-space are similar.

Making the reasonable assumption that the above statement has been proven, we select our particular geodesic as the intersection of two planes in K^3 :

(1) $q = \text{const.} = c_1 = l\sin\psi\cos\theta$;

(2) $s = \text{const.} = c_2 = l\sin\psi\sin\theta$

Dividing (2) by (1) gives the relation $\tan\theta = c_1/c_2 = c_3$.

***In other words, keeping θ constant defines a plane on K3*, which is clear anyway from the definition of spherical coordinates. But then θ is constant , then $d\theta = 0$, which means that $d\gamma = 0$, and γ can also be considered an arbitrary constant. This eliminates the difficulties of dealing with equation (3), and the equation of our "generic" geodesic in the projection of the conic surface into a hyper-plane of R^4 is**

$$\boxed{l\sin\psi = \frac{c_2}{\sin\theta} = \frac{c_2}{\sqrt{1+(c_2/c_1)^2}} = \frac{c_1c_2}{\sqrt{(c_1)^2+(c_2)^2}} = m},$$

where m functions as a kind of slope. Translating this into extrinsic coordinates gives the equation we are looking for:

$$\boxed{r\sin(h\beta) = m}$$

In particular, let us suppose that h has the value 1/4 . In the situation in 3-space this is associated with a vertex angle on the flattened surface of $\pi/2$. The equation is exactly the same as the one obtained by the projection of the conic surface K^2 onto the x-y plane, and describes a curve with a single self-intersection. Indeed at a point of self-intersection, one requires, in polar coordinates, that

$$\boxed{\begin{aligned}&\sin(\beta/4) = \sin(\beta + 2\pi/4) = \sin(\beta/4 + \pi/2)\\&= \cos(\beta/4)\\&\therefore \beta/4 = \pi/4, \beta = \pi\end{aligned}}$$

There is thus a self-intersection at the angle π .

Having reduced the 4-dimensional case to the behavior of geodesics on planes determined by g = const., the physical situation, in terms of Newtonian self-attraction is identical too that of the 2-cone surface in 3-space.

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