Introduction

Vive La Différence! (2) evokes the scientific world of the French Enlightenment. Mme. Emilie du Chatelet, mathematical prodigy, ardent Newtonian, Voltaire's protector and mistress, author of the earliest translation of Isaac Newton's *Principia* into French, was one of the central personalities in the great 18th century controversy over the relative merits of the models of the cosmos propounded by Descartes, Leibniz and Newton.

Newton's emergence as the scientific victor was due to the superiority of Newtonian mechanics for prediction and verification. The mathematics of statics and kinematics derivable from the Cartesian model gives incorrect answers, while the conceptions of Leibniz were, for their time, too general to develop any mathematics at all from them.

Philosophically the critiques of Descartes and Leibniz retain their merit down to our own times. The imprint of Cartesian thinking is clearly present in modern Quantum Field Theory, while Leibniz returns in full force in General Relativity.

In addition to Voltaire one counts among the beneficiaries of Emilie du Chatelet's generosity and hospitality a dazzling circle of scientists and philosophers, including Pierre Louis Moreau de Maupertuis, Charles Marie de la Condamine, Jean Mairan, Johann Samuel König, Johann Bernoulli, and Alexis-Claude Clairaut.

Therefore this study aspires to be more than merely a sketch the life of a brilliant woman of the French Enlightenment. It will also be dealing with the period, extending over half a century, in which the fundamental issues in mathematical physics and scientific philosophy for 3 centuries were first identified and debated .

The first 3 chapters of this book is a gallery of portraits, those of Mme du Chatelet, Voltaire and Maupertuis. The intricate, indeed intimate dynamics of their personal interactions can serve as a metaphor for their collective fascination with the mechanisms of the cosmic order.

A brief commentary on the contentions of Cartesians and Newtonians serves as an introduction to an original paper of the author's closely related to these matters : ***Trains and Fly*** **.**

Reasoning from first principles laid down by Aristotle, René Descartes had concluded that all dynamical behavior between material bodies resulted from collisions. This viewpoint followed logically from the assertion that what we call matter is nothing more than pure extension.

In other words , although in Descartes' writings one finds the first correct statement in print of the ***Galilean Principle of Relativity*** **of inertial motions and reference frames, they lead to calculations for the velocities of billiard balls after collision, based knowledge of their velocities before the collision, that are incorrect . The credit for understanding the right way to make these calculations, (essentially Newton's 2nd Law of Action and Reaction) , belongs to Christian Huygens.**

The Cartesians rejected Newton's theory of gravitation which depends on the existence of forces acting at a distance, a concept which

has no meaning in the Cartesian world-view in which "distance" is a derived quantity created by matter. As an interesting aside, in an attempt to reconcile Cartesians with Newtonians, Johann Bernoulli invented Fluid Mechanics.

After ***Trains and Fly*** **the history of quarrels between Leibnizians and Newtonians is continued in** ***The Great Imbroglio of 1750*** **. Their severity was such as to come very close to destroying the Prussian Academy of Science founded by Frederick The Big in 1748 . Like a Black Hole that sucks in all surrounding matter, (and returning little more than a faint trace of Hawking radiation) , their feuding dragged in some of the most eminent philosophers, mathematicians and physicists of the age. This chapter might also have been named :*****The Great Reaction to Least Action*** **!**

A short commentary on the Newton/Leibniz disagreement on the matter of Absolute versus Relative Space introduces my second paper: ***Conical Gravity*** **.**

The material in ***Vive La Difference (2)*** **is at the level of graduate students and advanced undergraduates in mathematics and physics. It is anticipated that many of its ideas and constructions will be insightful for scientists in general.**

Emilie du Chatelet

Emilie Le Tonnelier de Breteuil was born in the reign of the Hitlerian Sun King Louis 14 on December 17, 1706 to a well-to-do aristocratic family. Her aptitudes for the sciences were recognized at an early age and, although a woman, she was encouraged in them and treated with respect. Her gifts had been revealed in a traditional and obvious way through her childhood stunts as a calculating prodigy. Such prowess is not correlated with intelligence, and neither guarantees nor precludes the emergence of a mature scientist. In her case it did because her family had interpreted it as an indication that she had gifts worth developing.

Her mathematical education was furthered by a succession of capable tutors. Eventually these would come to include Johann Samuel Koenig, Pierre-Louis de Maupertuis and Alexis-Claude Clairaut.

In 1725 she married a military officer, the Marquis du Chatelet. For most of the 14 years of their marriage he was away in military campaigns in Poland and elsewhere, an arrangement which seems to have suited both of them. The Marquis seems to have accepted her numerous amorous intrigues with good grace, and did not take umbrage at her dedication, which bordered on obsession, to scientific study and research.

Mme du Chatelet's entrance onto the stage of the French Enlightenment occurs on April 25th, 1733, the date of her initial encounter with François Marie Arouet de Voltaire. It took place at the re-opening of a failed opera by Paradis de Moncrif "L'Empire de l"Amour", with a libretto thoroughly revised by Voltaire. He was well chosen for the task, having established a solid reputation in Parisian theatrical

circles from his play "Zaire", which had been a smash hit when produced in 1732.

From that time forth they were frequently seen in one another's company. The definitive event of their relationship occurred several months later, in the town of Montjeu in Burgundy. As will be recounted in more detail in the next chapter , Voltaire, owing to the publication of several manuscripts deemed offensive to both church and state, was in danger of being arrested and taken to the Bastille. In his long reign Louis 14 had made heroic exertions to drag France back into the obscurantism of medieval Catholicism. Although Louis 15 was somewhat more liberal (let us perhaps say libertine) , tolerating disreputable but harmless authors like the Abbé Prévost, his regime was not disposed to endure the sort of anti-clerical satire of which Voltaire is the acknowledged grandmaster.

Arriving at Montjeu Voltaire was warned by the compte d'Argental that a secret letter of denunciation, known as a *lettre du cachet* , had been issued against him. This meant that his arrest was imminent. Chatelet prevailed upon him to seek refuge in her estate at Cirey (in the Champagne region and outside direct crown jurisdiction). After a visit to Strasbourg he arrived there at the end of May, 1734. Cirey would serve as his home base until Emilie du Chatelet's death in 1749 .

All through the following year Voltaire maintained a low profile . He had good reason to: in Paris on June 10, 1734, the entire French edition of his *Philosophical Letters* , (in which he praises everything English, disparages everything French, and attacks the Jesuits) , was publicly torn to shreds and burned. Voltaire was finally given an official

pardon in March 1735 and he and Chatelet returned to the capital for a visit.

One of the immediate consequences of Voltaire's residence at Cirey was the successful conversion of Chatelet from Leibnizian to Newtonian. Influenced by her teacher, Samuel König, she'd written a long essay, actually a physics textbook, entitled *Les Institutions de Physique* **, a defense of Leibnizian physics based on the** *Monadology* **and the application of the** *Principle of Sufficient Reason* **. In her formulation of the principle one hears an echo of the notion of the** *broken symmetry* **which plays an important role in modern particle physics:**

" *Il n'y'a personne qui se determine a une chose plûtot qu'à'une autre sans une raison suffisante qui lui fasse voir que cette chose est préferable à l'autre* **."**

(No individual entity choses to be one way rather than another, without there being a sufficient reason why that decision is not done in preference to all other possibilities.) One should keep in mind that in Leibniz's system all things are composed of monads, and that monads are living beings capable of reason.

However Voltaire, initially Cartesian, had met Maupertuis in Paris in 1732, who'd managed to convince him of the overwhelming superiority of Newtonian mechanics. Eager to imbibe the message of Newtonianism at its source, Mme du Chatelet hired Maupertuis in 1733 as her teacher in mathematical physics at Cirey . She eventually became the leading authority in Newtonian mechanics in France, though in the meantime she would fall desperately in love with Maupertuis. No-one in this circle seems to have attached much value to marital or amorous fidelity, but it is clear that Maupertuis did not reciprocate her feelings.

This situation, which Voltaire seems to have accepted for the moment ,was one of the contributing factors for the great animosity that emerged between Voltaire and Maupertuis at the Prussian Academy in the 1750's . René Vaillot writes (all references are to the Bibliography) :

" *On a pu taxer d'indelicatesse vis-a-vis de Voltaire la conduite d'Emilie avec Maupertuis* . "

(One might accuse Emilie of a lack of consideration vis-a-vis Voltaire in her dealings with Maupertuis.)

In 1739 Samuel König publicly accused Chatelet of having plagiarized his views on infinitesimals, [1] **in the** ***Institutions de Physique*** **. She appealed to Maupertuis, who defended her in a speech at the French Academy of Sciences. Praise for the** ***Institutions*** **also came from the Bernoullis, Clairaut and Mairan.**

On January 10, 1739, a few months after returning from his expedition to the Arctic Circle (to be discussed in its proper place) , Maupertuis visited Cirey and stayed for 3 days. The predictable disaster signaled the end of the Chatelet/Maupertuis involvement. After his departure she continued her instruction in mathematical physics with Clairaut.

Both Chatelet and Voltaire were workaholics. He wrote and studied through the day, she through the night. They survived on limitless amounts of coffee and only slept a few hours each day.

"*Elle travaillent surtout la nuit, elle ne dort que quelques heures et date souvent ses lettres de deux ou trois heures apres minuit.* " **(Vaillot)**

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[1] **which are actually those of Leibniz. Conveniently, Leibniz was not alive at the time to accuse König of plagiarism.**

(She works through the night, sleeps only a few hours and often dates her letters at 2 or 3 A.M.)

Her translation of Newton's *Principia* was published in August, 1737. Anticipating Subramanyan Chandrasekhar's English translation over two centuries later , she included an "Algebraic Appendix" to assist readers in understanding it.

In 1738 , without informing one another, Chatelet and Voltaire sent independent entries to a competition announced by the French Academy for the best essay on the nature of fire :

" *She kept her entry from Voltaire and completed it at night, sleeping an hour or so each day and keeping herself awake by plunging her hands in ice water.* " Osen (pg. 59)

It indicates no disparagement to them that the prize went to the greatest applied mathematician in history, Leonhard Euler. Neither philosophy nor mathematics were adequate to the task, and the understanding of the true nature of fire had to await the discovery of oxygen by Joseph Priestley and Antoine-Laurent Lavoisier in the latter part of the century.

Mme Emilie du Chatelet died in giving birth to the child from her lover in fashion at the time , the poet St. Lambert, on September 10th, 1749 . As we know, death from childbirth was a common occurence at the time, and made no distinction between the most ordinary housewife and and a daughter of the aristocracy who happened to be one of the major intellects of the age. Her son was destined to die at the guillotine in the Reign of Terror.

François Marie Arouet de Voltaire

Following his intemperate response to the public humiliation given him by the duc de Rohan in 1726 , Voltaire was briefly imprisoned in the Bastille. As part of the condition of his release, he was forbidden to reside less than 50 leagues from Paris. He satisfied the terms of his exile by going to England.

Political relations with England, never very good at the best of times, had recently been ameliorated by a treaty between the two countries negotiated by the duc d'Orleans. Following the death of Louis 14 in 1715 Orleans was appointed Regent for the young prince Louis 15 . The situation was politically charged because of marriages and alliances cemented in the previous reign. Had Louis 15 died prematurely, Philip 5 of Spain, Louis 14's grandson through Madame de Maintenon , could have asserted a claim to the French crown. The War of the Spanish Succession (1702-1714) had already been fought over this matter. d'Orleans negotiated the alliance with England to prevent the rekindling of this conflagration. It lasted until 1730.

While in England Voltaire met Pope, Arbuthnot, Swift, Gay. He read Gulliver's Travels and the script of the Beggar's Opera, and attended several performances of Shakespeare's plays which he didn't like because he didn't understand them. Friendly with prominent Newtonians he did not himself become one until 1732. Still, though he didn't give a fig for Newton, he is the inventor of the Newton's apple story.

In 1728 Voltaire's long poem *La Henriade* eulogizing Henry 4, the Huguenot king of France, was published in England. Under Papal ban in all Catholic countries, it was avidly subscribed to throughout Protestant Europe and gave him a good financial return.

His exile was lifted in April 1729 and he returned to France. Before that , in 1728 , he and the mathematician Charles Marie de la Condamine had worked out a scheme to defraud the Paris lottery. The science of economics was in its infancy and they were able to get away with it. Voltaire had no more financial worries for the rest of his life.

Money isn't everything. Within the first few years of his return to France 3 books of his were published that got him into serious trouble with the ruling reactionaries.

The *Temple du Gout* , 1731, is a satiric putdown of French writers which may have been modeled on Alexander Pope's *Dunciad* (1728) .

Next: part of the edition of his biography of the tyrant Charles 12 (1731) was seized and destroyed . The remaining copies were dispatched from Rouen to Versailles by his publisher Claude Jore , then smuggled into Paris from there in the private carriage of the duc de Richelieu.

The *Garde de Sceaux* , (Guardian of the Seal, a crown prosecutor) , Chauvelin, warned Voltaire that if he tried to publish *Les Lettres Philosophiques* in France, he would be severely punished. It had already been published in England in 1733, as *Letters Concerning the English Nation* . Without Voltaire's consent or knowledge, Claude Jore had already printed up 2,500 copies of the book. These were hidden in different places in France. Voltaire had also unwisely lent a copy to the Parisian printer, François Jossé , who used it as the basis of a pirated edition published in 1734! The rest of the story has been recounted above. There is a sequel.

As the outcome of a lawsuit brought by Claude Jore against him in 1736, Voltaire was fined 500 francs to be given "to the poor". He'd gotten Jore into trouble by revealing himself as the author of *Les Lettres*

#11...

Philosophiques .

Maupertuis

Pierre-Louis Moreau de Maupertuis (1698-1759) is one of the truly fascinating figures in European science of the 18th century. Although he did not achieve distinction in any one field, the extent and diversity of his ideas and activities place him among the great scientists of any age.

The initial reception to the publication of his treatise in defense of Newtonian mechanics, *Discours sur les differantes figures des astres* (1733), aroused the hostility of the Cartesians. Not only was it a matter of loyalty to France's most celebrated natural philosopher, but many people were sincerely troubled by the notion of gravity as a force acting at a distance. Even after General Relativity this problem cannot be said to have been solved.

In due time this treatise, together with the work of Chatelet, Voltaire, Euler and the Bernoullis, would lead to the triumph of Newtonianism, laying the ground for the great advances in the mathematics of Celestial Mechanics of Lagrange, Laplace and, in a later age, Poincaré .

Though essentially philosophical , Descartes system of the world does lead to predictions which can be falsified. The grinding action of the refined matter that creates the vortices that move the planets along their orbits, would have the effect of flattening the Earth at the Equator. But according to Newtonian mechanics, the Coriolis force caused by the Earth's rotation not only predicts a flattening at the poles but allows one to compute the extent of the flattening.

In 1737 an expedition organized by astronomers La Condamine, Godin and Bouger traveled to Peru to measure the length of a degree of

latitude at the Equator. It encountered numerous difficulties and was unable to return for 11 years! Shortly before their expedition Maupertuis and Anders Celsius had gone to Lapland to measure the size of a degree of latitude at the Arctic Circle. Despite the general hostility of the French to English science, it was acknowledged that the best makers of scientific instruments were in England, and Maupertuis and Celsius had managed to obtain the latest English instrumentation for their measurements. This expedition lasted a bit more than a year.

By all the evidence Maupertuis had a hell of a time in Lapland. A year after his return to France in November 1737, a pair of Lapp sisters arrived in the French capital, claiming that Maupertuis had married both of them! Fabulous accounts of their exotic character circulated fashionable Paris although in fact they were the daughters of a middle-class Swedish merchant. Eventually they were given small pensions by the French government on the condition that they convert from Lutheranism to Catholicism. One of them entered a convent, the other made an unfortunate marriage culminating in a series of lawsuits against her husband.

Following their presentation at the French Academy , Maupertuis' findings were published in a brilliant book entitled *Examen désinteress é des différents ouvrages qui ont été faits pour déterminer la figure de la Terre* **(1738). Maupertuis, Chatelet, Voltaire and all the French Newtonians were viciously attacked by the Cassini family of astronomers, the fanatical Cartesian Jean Mairan (1678-1771), the famous naturalist René-Antoine Réamur and others. All dust against the Mistral: the superiority of Newtonian mechanics could no longer be denied and the hey-day of Cartesianism in its country of origin was over. As will be**

seen , an equally futile rear-guard action was mounted in Germany by the Leibnizians in the 1750's .

In 1758, Maupertuis was invited by the mechanist philosopher de la Mettrie and the Prussian ruler Frederick the Big to come to Berlin and re-vitalize the moribund Prussian Academy of Sciences. He acquitted himself of this task with characteristic brilliance. Within a few years he'd managed to lure many of the best minds in Europe to the dreary Prussian capital, including Condillac, the students of the greatest medical researcher of the 18th century, Albrecht Haller (1708-1777) , Leonhard Euler, Samuel Konig. and Voltaire himself.

The Principle of Least Action

For two and a half centuries, the Principle of Least Action in various formulations, has been the fundamental idea underlying all theoretical mechanics, both classical and quantum. It is unfortunate that Maupertuis tried to apply the Principle of Least Action to a proof of the existence of God. Not only were the arguments absurd, it brought him directly into the line of fire of the Leibnizians at the Academy who regarded the presence of the French upstart with suspicion.

Maupertuis confuses action integrals stationary over time with those stationary over length. In fact the constraining quantity in the Maupertuis principle is energy. The confusion was straightened out by Euler, who gives full credit to Maupertuis for opening up the connection between mechanics and the calculus of variations.

The Principle of Least Action has two forms:

(1) ***Maupertuis' Principle*** **: Along an actual path followed by a particle in a conservative field of force, the total** ***Maupertuis action*** **is a relative extremum. More precisely it is an extremum when contrasted**

with the actions that would be developed by the particle if it moved with the *same total energy* along any one of the *neighboring* constraining paths between the same terminal points A and B (d'Abro, Vol I, pg. 262)

(2) *Hamilton's Principle* : If a free particle moving in a conservative field of force is thrown from a point A and reaches a point B, the path actually followed , when contrasted with all other neighboring constrained paths (extending between A and B), which would be covered *in the same interval of time* , will be such that, along it, the *Hamiltonian* is an extremum or, if we prefer, is stationary. (d'Abro, pg, 263)

The publication of the *Cosmologie* led to the "great imbroglio" described in a following section. In his defense of Maupertuis' claim to be the discoverer of least action, Euler wrote:

"*The equilibrium principle is not only perfectly stated, but leads one all by itself to all of the investigations that have been made in statics and dynamics, with the result that by means of this single principle, the whole science of equilibrium can be explicated, with no need of any other principle* "...

"*Therefore it is the case that the principle of Maupertuis fulfills this function, one can assert without any doubt that it not only comprises the essence of all our knowledge of equilibrium, and not only provides the very basis of that science, but must be deemed the most sacred law underlying all of Nature ...* "

In 1751, Euler suggested that Maupertuis' principle could be extended to Elasticity. To d'Alembert, the merit of the Maupertuis principle, (closely allied to his concept of 'virtual work') was that it quantified the vague commonplace that "Nature uses the most

economical means to arrive at its goals."

Cartesians versus Newtonians
Some comments

In contrast to Galileo the philosophical and scientific system devised by René Descartes should be understood as an extension rather than a refutation of the ideas of Aristotle. Starting from the dogma of total separation of mind and body Descartes argues that, since mind substance lacks spatial extension, while material objects are perceived as manifestations of extension, all the characteristics of matter are nothing more than those of pure extension.

Extended objects, objects taking up space may collide, and when they do they repel one another. Since matter is nothing more than extension, Descartes concludes that all dynamical change in the universe is caused by collisions. There can be no place in this scheme for action at a distance, nor for a principle as basic as the Conservation of Energy. Only the Conservation of Momentum is allowed, which is why Descartes, though proclaiming a billiard ball universe, could not calculate the paths and velocities of billiards after collision!

On the basis of another Aristotelian notion , " Nature abhors a vacuum" , Descartes claims that there is no such thing as empty space. Instead one finds a highly refined interstellar dust that forms into swirling vortices moving the planets along their orbits.

Descartes' ideas were incorrect of course, but his manner of proceeding conceals a deeper issue that does not disappear with their refutation. The problem is one of the methodology of science: in drawing up a picture of the nature of the world, how much weight does one assign to observation and experiment and how much to reasoning from

first principles ? Whether separate or intimately involved, both substance and mind are present in the world as we know it . Any claim to understanding its essential nature must include components of empirical science and philosophy. Throughout scientific history serious errors have occurred through the insistance of an exclusive focus on one or the other.

We now know that the atomic theory of Democritus, Empedocles, Epicurus and Lucretius was, in its essentials, correct. It was arrived at entirely by deduction from first principles, yet was neglected for millennia , until the 19th century, because there was no concrete evidence to support the existence of atoms, molecules or elementary particles.

Likewise, the cardinal error of the French Cartesians was to assert that , because the first principles of Descartes do not lead to the possibility of action at a distance the theories of Kepler and Newton that accurately predicted the functioning of the solar system had to be wrong.

The chauvinistic rivalries of English Newtonians, French Cartesians and German Leibnizians are of little interest today. Once Newtonian mechanics was accepted, French and Swiss mathematicians, Lagrange, Euler, the Bernoullis, Laplace and others, elaborated its consequences far beyond anything the English would be capable of doing for another century. Cartesianism has returned to fundamental physics through the Quantum Field Theory of Dirac, which posits the creation of virtual particle pairs from the zero point energy of empty space, the Quantum Electrodynamics of Richard Feynman, and the contemporary search for Dark Matter.

#19...

And Leibniz emerges triumphant in General Relativity, a theory created by a man who fled Germany with a price on his head.

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Trains and Fly

From an initial displacement of 100 miles of track, two trains are moving towards one another at 50 mph. A fly travels between them. The fly begins from the left at time 0, with an initial velocity of 90 mph. . The tracks are frictionless, there is no air resistence, and no source of energy. All of the dynamics are due to momentum exchange in collisions.

The mass of each train is M. The mass of the fly is ε . These values M and ε have been obtained from sources on the Internet (see Bibliography :

Weight of 200 flies = 0.22 grams

1 flyweight =0.0011 grams ~ 10^{-3} grams

For a train with tender of class CSA-1:

Weight = 876,000 lbs. = $\boxed{\dfrac{876,000}{2.21}\times 10^{3}\,grs.}$

Let q =ε /M .

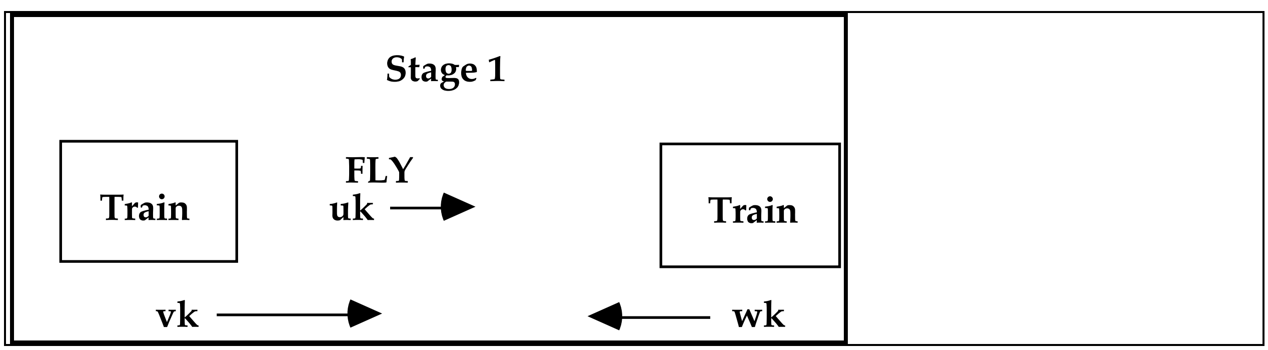
Then

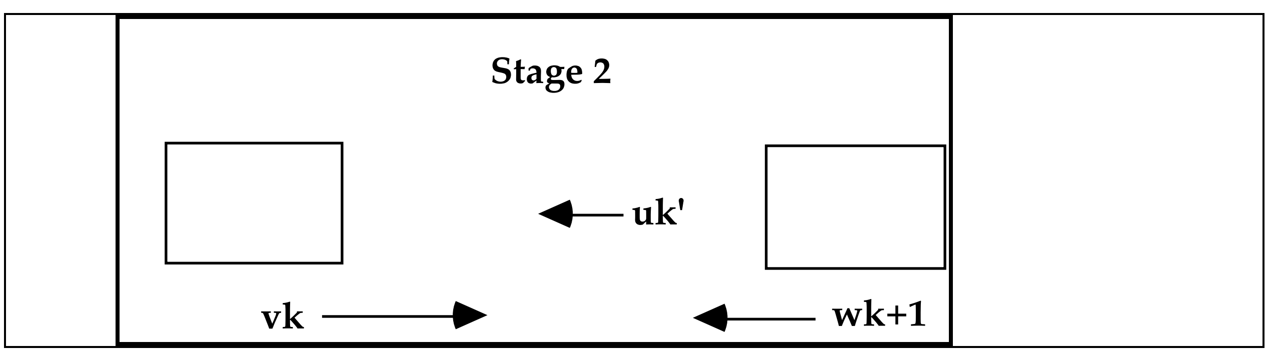
$$\boxed{\begin{aligned} q &= \left(\frac{876,000}{2.21}\times 10^{3}\right)^{-1}\times 10^{-3} \\ &= \frac{2.21}{876}\times 10^{-9} \approx 0.25\times 10^{-11} \end{aligned}}$$

That is to say, the train is of the order of 4x100 billion times heavier than the fly.

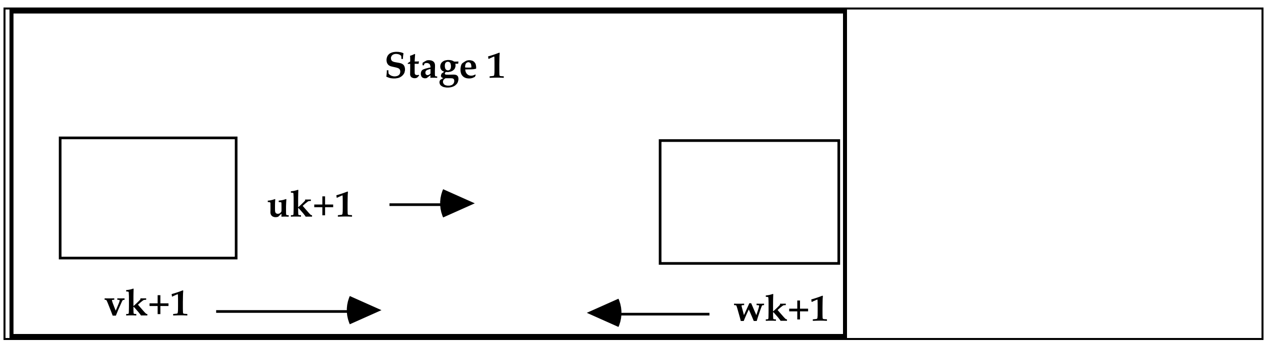
The total distance will not figure into this problem (unlike the classical von Neumann " trains and fly" mathematics problem which uses no physics). We are interested in the energy and momentum exchanges under the restrictions of the conservation of energy and conservation of momentum. The fly goes from left to right, collides with the right-hand train, is bounced back with higher momentum and collides with the left train, sending it off again from left to right. This entire process is treated as a single cycle , indexed by k = 1,2,3..... The two stages are clearly presented in the following diagrams:

Cycle k





Cycle k+1



In stage 1 of cycle k , the fly moves from left to right with velocity u_k to collide with the train coming towards it with velocity w_k . As a result of their collision the fly moves from right to left with velocity u_k' . the train on the right has acquired a new velocity w_{k+1} .

When the fly next collides with the train on the left, its velocity will be altered from u_k' to u_{k+1}, while the velocity of the train on the left will be changed to v_{k+1} . This initiates the next cycle .

The equations corresponding to a complete cycle are given below:
Let J = Total Momentum at the beginning $= 90\,\varepsilon + 50M - 50M = 90\varepsilon$.

E = Total Energy at the beginning $= 1/2\,(\,(90)^2\,\varepsilon + 2(50)^2 M\,)$ Then:

$$\boxed{\begin{aligned}&(i) M(v_k + w_k) + \varepsilon u_k = J\\&(ii) M(v_k^2 + w_k^2) + \varepsilon u_k^2 = 2E\\&(iii) M(v_k + w_{k+1}) + \varepsilon u_k' = J\\&(iv) M(v_k^2 + w_{k+1}^2) + \varepsilon (u_k')^2 = 2E\\&(v) M(v_{k+1} + w_{k+1}) + \varepsilon u_{k+1} = J\\&(vi) M(v_{k+1}^2 + w_{k+1}^2) + \varepsilon u_{k+1}^2 = 2E\end{aligned}}$$

A quick examination shows that we have 6 equations in 7 unknowns. After simplifying this set of equations we will select one variable from which all the rest may be derived.

In stage one of the momentum transfer, the velocities u_k and w_k are transformed into u_k' and w_{k+1} . It is important to keep in mind that the positive direction is from left to right and that, in the initial stages at least, w_k is a negative number. Classically the momentum transfer is given by:

$$\boxed{\begin{aligned}&\mu = \frac{2M\varepsilon(w_k - u_k)}{M+\varepsilon}\\&(1)\ u_k' = (u_k + \mu/\varepsilon) = (u_k + \frac{2M(w_k - u_k)}{M+\varepsilon})\\&= \frac{2Mw_k - u_k(M-\varepsilon)}{M+\varepsilon}\\&(2)\ w_{k+1} = w_k - \mu/M = \ldots = \frac{2\varepsilon u_k + w_k(M-\varepsilon)}{M+\varepsilon}\end{aligned}}$$

Since u_k is initially very tiny, and M is huge compared to ε , (by a factor of 10^{11} !) , the collisions of stage 1 ***result in an approximate net increase of twice the velocity wk to the velocity of the fly*** **, while the alteration of the train's velocity is negligeable. This state of affairs will change after a great many collisions occured .**

The collision in stage 2 involves the returning fly and the train on the left:

$$\mu' = \frac{2M\varepsilon(v_k - u_k')}{M+\varepsilon}$$

$$(3)\ u_{k+1} = u_k' + \mu'/\varepsilon = u_k' + \frac{2M(v_k - u_k')}{M+\varepsilon}$$

$$= \frac{2Mv_k - u_k'(M-\varepsilon)}{M+\varepsilon} = \frac{2Mv_k - (M-\varepsilon)[\frac{2Mw_k - (M-\varepsilon)u_k}{M+\varepsilon}]}{M+\varepsilon}$$

$$=\ldots=(\frac{2M}{M+\varepsilon})v_k - \frac{2M(M-\varepsilon)}{(M+\varepsilon)^2}w_k + (\frac{M-\varepsilon}{M+\varepsilon})^2 u_k$$

$$(4) v_{k+1} = v_k - \mu'/M = v_k - \frac{2\varepsilon}{M+\varepsilon}(v_k - u_k')$$

$$=\ldots=(\frac{M-\varepsilon}{M+\varepsilon})v_k - \frac{2\varepsilon(M-\varepsilon)}{(M+\varepsilon)^2}u_k + \frac{4M\varepsilon}{(M+\varepsilon)^2}w_k$$

It is possible to combine these equations with the expressions for momentum and energy so as to express every variable as a function of u$_k$

.

Rewrite the conservation equations (i) , (ii) , as

$$Mw_k = J - \varepsilon u_k - Mv_k$$

$$(ii) M(v_k^2 + (\frac{J - \varepsilon u_k - Mv_k}{M})^2) = 2E - \varepsilon u_k^2$$

Expanding the left side and transposing:

$$\boxed{\begin{aligned}&Mw_k = J - \varepsilon u_k - Mv_k\\&M(v_k^2) + (\frac{J - \varepsilon u_k - Mv_k}{M})^2)\\&= 2E - \varepsilon u_k^2\\&Mv_k^2 + 1/_M(J^2 + \varepsilon^2 u_k^2 + M^2 v_k^2 - 2J\varepsilon u_k - 2JMv_k + 2\varepsilon M u_k v_k)\\&= 2E - \varepsilon u_k^2;\\&\ldots\\&2M^2 v_k^2 + v_k\{2M\varepsilon u_k - 2MJ\}\\&+[\{u_k^2 \varepsilon(M+\varepsilon) - 2J\varepsilon u_k\} + \{J^2 - 2ME\}] = 0\end{aligned}}$$

This is a quadratic in the variable v_k . Its coefficients are polynomials in u_k

$$\boxed{\begin{aligned}&P(x) = \{2M\varepsilon x - 2MJ\}\\&Q(x) = x^2\varepsilon(M+\varepsilon) - 2J\varepsilon x + (J^2 - 2ME)\end{aligned}}$$

As equations (i) and (ii) are symmetrical in v_k and w_k , they appear as the two solutions of the quadratic, and are given by:

$$\boxed{(A){:}\ v_k, w_k = \frac{-P(u_k) \pm \sqrt{P^2 - 8M^2 Q}}{4M^2}}$$

Since ε(M+ε) is infinitesimal at the beginning of the process, Q(x) will be minus. Therefore, initially v_k is the positive solution, w_k the negative.

Finally, u_{k+1} may be expressed as a function of u_k by virtue of equation (3) above. Note once again that u_k is always positive, with u_k' being the velocity in the negative direction.

THEOREM:

The speeds of the trains at every cycle are a function only of the velocity of the fly. Likewise the velocity of the fly in each cycle can be computed as a function of the velocity of the fly in the previous cycle.

For many many thousands of hits one can assume that $\varepsilon = 0$. Eventually however:

$$\boxed{\begin{array}{l} P(u_k) \approx 0; Q(u_k) \approx -2ME \approx -5000M^2 \\ v_k \approx 50; w_k \approx -50 \\ u_{k+1} \approx 2v_k - 2w_k + u_k = 200 + u_k; \\ \therefore u_n \approx u_0 + 200n = 200n + 90 \end{array}}$$

In the initial phase, the velocity of the fly increases by 200 mph with each cycle. Over time the size of the *increments* **will diminish, although the velocity of the fly will** *continue to rise until the trains are arrested and turned back by the fly* **.**

Without going into the details one is able, from the above equations, to estimate the value of u_n at which v_n and w_n are closest to 0. Then $Q(u_n)$ will be approximately equal to 0 , or

$$\boxed{\begin{array}{l} Q(u_n) = u_n^2 \varepsilon (M + \varepsilon) - 180\varepsilon^2 u_n + (J^2 - 2ME) = 0 \\ J = 90\varepsilon; E = \frac{1}{2}((90)^2 \varepsilon + 2(50)^2 M) \end{array}}$$

This is another quadratic, in u_n , with (positive) solution

$$\boxed{u_n = \frac{180\varepsilon^2 + \sqrt{(180\varepsilon^2)^2 + 4\varepsilon(M+\varepsilon)(2ME - J^2)}}{2\varepsilon(M+\varepsilon)}}$$

Ignoring the tiny amount at the left of the right-hand side, one gets:

$$\boxed{\begin{aligned} u_n &\approx \frac{\sqrt{(180\varepsilon^2)^2 + 4\varepsilon(M+\varepsilon)(2ME - J^2)}}{2\varepsilon(M+\varepsilon)} \\ &= \sqrt{\frac{(180\varepsilon^2)^2 + 4\varepsilon(M+\varepsilon)(2ME - J^2)}{4\varepsilon^2(M+\varepsilon)^2}} \\ &= \sqrt{(\frac{90\varepsilon}{M+\varepsilon})^2 - \frac{(90)^2\varepsilon}{M+\varepsilon} + \frac{2ME}{\varepsilon(M+\varepsilon)}} \end{aligned}}$$

The first two terms in this expression are negligible. Thus u_n is approximately equal to :

$$\boxed{\begin{aligned} u_n &\approx \sqrt{\frac{2ME}{\varepsilon M}} = \sqrt{\frac{2E}{\varepsilon}} \approx \sqrt{\frac{2500M}{\varepsilon}} \\ &= 50\sqrt{\frac{M}{\varepsilon}} = \frac{50}{\sqrt{q}}! \end{aligned}}$$

As calculated at the beginning of this paper, q is of the order of 10^{11}. Therefore, the peak speed which the fly must obtain before the process is reversed is about

$$\boxed{S \approx 50 \times \sqrt{10} \times 10^5 = 5 \times \sqrt{10} \times 10^6\, mph}.$$

As there are 3600 seconds in the hour:

$$\boxed{S \approx \frac{3 \times 10^7}{7200} mps = \frac{10^5}{24} mps}$$

an enormous figure but still far short of the speed of light.

Qualitative Analysis

PHASE I: In the initial phase the weight and momentum of the fly may be considered negligible. In each of many thousands of recoils, its velocity augments by about 200 mph. As it increases its speed the trains are slowed down infinitesimally by collisions.

PHASE II : As the trains slow down, the amount by which each recoil *increases* the velocity of the fly is also reduced. At the same time however, *the amount by which the velocity of each train is reduced increases*. Look at equation (4):

$$\boxed{(4)\, v_{k+1} = \left(\frac{M-\varepsilon}{M+\varepsilon}\right)v_k - \frac{2\varepsilon(M-\varepsilon)}{(M+\varepsilon)^2}u_k + \frac{4M\varepsilon}{(M+\varepsilon)^2}w_k}$$

Over time u_k increases enormously, v_k and w_k go down. Thus the middle term predominates. Until the trains begin to reverse their direction, u_k cannot decrease. This is confirmed by equation (3):

$$\boxed{(3)\, u_{k+1} = \left(\frac{2M}{M+\varepsilon}\right)v_k - \frac{2M(M-\varepsilon)}{(M+\varepsilon)^2}w_k + \left(\frac{M-\varepsilon}{M+\varepsilon}\right)^2 u_k}$$

***Observe that all three terms on the right hand side are positive* !**

The velocity of the fly peaks at its maximum when the polynomial $Q(u_n) \sim 0$. This has been shown to be in the neighborhood of $\boxed{u_n \approx \frac{50}{\sqrt{q}}!}$, which for this particular problem is about 30 million mph, an enormous figure but well below the speed of light. One must keep in mind the fact that the speed of each train become extremely minute in the critical change-over stage , and that the fly will bat away at a speed very close to this maximum for thousands of cycles.

PHASE III : At the critical point at which the fly begins to *reverse* the direction of the trains, the velocities vk and wk *are still governed only by the velocity of the fly, with the + and - signs reversed*. In other words, if the velocities of the left and right trains respectively at some forward cycle are about V and -W, while that of the fly is U, then in the reverse

cycle the velocities of these trains will be about -W and V, when the velocity of the fly arrives once more in the neighborhood of U. This follows from the conservation of momentum and energy.

PHASE IV: As long as the trains are moving away from each other at velocities less than 50 mph, the fly's velocity will stay above 90 mph. However something very strange happens when the train velocities are slightly greater than or equal to 50. By the conservation of energy, (and also because the velocity of either the fly, left train or right train is sufficient to determine all the others), the fly's velocity will fall to below 90 mph .

When this happens, the fly will strike either the right or the left train one more time , then recoil with a velocity less than or equal to 10 mph.

In the final phase, the 2 trains continue on forever moving away from each other at about 50 mph, while the fly trails along either to the right or the left, at about 10 mph. The system is neither periodic, nor does it die away in a whimper. Finally we will show that , at every cycle of this process the difference $|v_k+w_k|$ remains very tiny. By the conservation of momentum, one has:

$$\boxed{\begin{aligned} &M(v+w)+\varepsilon u = 90\varepsilon; \\ &v+w=\frac{\varepsilon(90-u)}{M}=q(90-M) \end{aligned}}$$

The previous estimate on the maximum value of u is $\boxed{u \approx \frac{50}{\sqrt{q}}}$

Hence

$$\boxed{\begin{array}{l} |v+w| \le 90q + 50q\sqrt{1/q}; \\ q \approx \dfrac{1}{4\times 10^{11}}; \\ 90q + 50\sqrt{q} \approx \dfrac{90}{4\times 10^{11}} + (50\sqrt{10})10^{-5} \\ \approx (150)10^{-5} \end{array}}$$

Therefore the difference between the speeds of the two trains under the relentless pounding of the fly will always be of the order of (1.5x10^{-3} mph.)

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The Principle of Least Action and the Great Imbroglio of 1750

Leibniz, Voltaire, Maupertuis, König, Euler , Johann Bernoulli, Frederick the Big and the Prussian Academy

In 1748 the philosopher of the clock-work soul, Julien Offroy de La Mettrie (*Man The Machine* , 1748) suggested to Frederick the Big of Prussia that Pierre-Louis Moreau de Maupertuis be invited to Berlin to reorganize the Prussian Academy of Sciences . The Academy had filled up with retired medical doctors. Shortly after his arrival the academicians asked him to oversee the composition of an "Encyclopedia of Metaphysics". Maupertuis nixed the project on the grounds that it would make the Academy look ridiculous. Several of them were senile.

After Emilie du Chatelet's death in 1749, Maupertuis was able to tempt Voltaire to come to Berlin. Around the same time he also brought

on board Chatelet's old math teacher, Johann Samuel König. In March 1751 König published a memoir in the Academy's professional journal, the *Acta Eruditorium* , filled with strange and cranky ideas. True to form, he now accuses Maupertuis of plagiarism not, this time, from himself, but from the incomparable Leibniz! König asserted that the *Principle of Least Action* enunciated in Maupertius' treatise on Cosmology, had been stated verbatim in a letter Leibniz had written on October 16th, 1707 to a correspondent in Switzerland, Captain Henzi.

Tragically Henzi was not longer available to confirm König's accusations. He'd been beheaded by the Swiss government in the aftermath of the suppression of a *coup-d'état* of which Henzi had been the leader. The Prussian Academy asked König if this letter might be among Henzi's effects, still in the keeping of the Swiss government, and insisted that he make an effort to track it down. It is now generally believed, (though it has not been proven) , that König invented the story of the letter. Such behavior was, unfortunately, fairly common among scientists in the 18th century: the grand master of plagiarism accusations was not himself immune from the charge of fabrication. Anxious to spare König further humiliation Formey, the secretary of the Academy, tried to persuade Maupertius to stop the investigation. The proposal threw Maupertuis into a rage.

König placed himself at the head of a clique of Leibnizians out to vilify Maupertuis. They criticized Maupertuis for his views on infinitesimals. They derided his belief in the possibility of solid bodies. Some proclaimed that "force" had to be more fundamental than "action", while the theologically minded were contemptuous of his "proof of God via the Principle of Least Action". The physical

teleologists on the other hand thought that Maupertuis had not gone far enough!

No less an authority than Leonhard Euler was invited to come to the Prussian Academy to settle the dispute. This he did, on April 13, 1752. He defended the Principle of Least Action with all the brilliance at his command, ascribing all the credit to Maupertuis. Euler finished his lecture with a condemnation of König's behavior, which Maupertuis tried to use as grounds for impeachment proceedings. Before this could happen König resigned.

In a startling about-face, the followers of König now accused Maupertuis of having stolen the *Principle of Least Action* from Euler ! This was refuted by Euler himself. Down but definitely not out, König brought in an accusation against Euler in September of 1752, for having dared to speak at the Academy without proper authorization! Accusing Maupertuis of having stolen the Principle of Least Action from *both* Leibniz and Euler, he argued that it was false anyway ! König continued to pile up malicious slanders against Maupertuis, to which he considered beneath his dignity to respond.

Soon afterwards Euler published 2 papers laying the foundations of the *Calculus of Variations* and *Euler-Lagrange* mechanics. Once again crediting Maupertuis with its discovery, he derives the *Principle of Least Action* directly from Newton's laws and self-evident assumptions .

Up to that point it looked as if Maupertuis was winning; but this was the moment when Voltaire jumped into the fray.

Friction between Maupertuis and Voltaire had ignited from the day of his arrival in Berlin. Voltaire seems to have resented the fact that

Maupertuis had replaced him as the French intellectual darling of the Prussian court. There may also have been some lingering jealousy from the brief infatuation of Chatelet for Maupertuis. Internationally renowned as the arch foe of every word ever to issue from the pen of Leibniz, Voltaire suddenly presented himself as principal as adversary to the Newtonianism of Maupertuis!

Libelous, ugly and stupid attacks on Maupertuis issued from Voltaire's pen in a steady stream. Not the least of his slanders was the claim that Maupertuis was psychotic. These polemics were later gathered up and published in book form under the title *The Adventures of Dr. Akakia* : Dr. Shithead in other words . Voltaire became so abusive that Frederick the Big had no choice but to dismiss him from the Academy and send him back to France. Although Maupertuis died in Switzerland in 1759 , Voltaire continued to defame him in an thoroughly ignoble fashion until his own death in 1778 at the age of 84.

Newton, Leibniz and Absolute Space

Newton recognized that his arguments in defense of the existence of an Absolute Space (outside of all reference frames yet identical in all its properties when seen from any one of them) were inadequate. He himself proposed a thought experiment which appears to cast doubt on the possibility of Absolute Space:

Imagine that at the same moment every object in the universe were to be given an identical acceleration in a specified direction. Would anyone be able to detect such an acceleration?

Newton claims that this phenomena is detectable relative to the fixed stars. His two basic arguments for claiming that physical location in space is frame independent are :

(1) The immobility of the constellations, or fixed stars; and

(2) The pail of water experiment.

If a pail be filled with water, then twirled rapidly around its axis of symmetry, the water will climb the sides of the pail and empty out at the base. This effect is independent of both the presence of gravitation and of air friction and implies the existence of an absolute external reference frame. Since such a frame cannot be deduced from local phenomena, it must be present in the frame of the constellations, which has never been observed to alter since the earliest astronomical observations in Antiquity.

The argument is a sound one, based on substantial considerations. General Relativity solves the paradox to some extent, although questions remain. According to GR, all motion in a gravitational force field is completely relative. Yet the equivalence of gravitational mass and inertial mass is the cornerstone of General Relativity, and the inertial properties of matter are determined by the distribution of matter throughout the whole universe. This assertion is called *Mach's Principle* . If one replaces the phrase "the fixed stars" in Newton's *Principia* by "Mach's Principle", Newton's water pail experiment carries over into General Relativity.

"Mach's Principle" is the weakest assumption in the theory of General Relativity. It is essentially heuristic and cannot be tested by any experiment. It is the concerns of Gottfried Wilhelm Leibniz which find their profound reflection in General Relativity's philosophical foundations . In it one finds Leibniz's *Principle of Sufficient Reason* ,

Principle of the Identify of Indiscernibles **, aspects of the** *Monadology* **, and his arguments against Newton's Absolute Space.**

For Leibniz the *Principle of Sufficient Reason* **:**

"*Nothing happens without a reason why it should be so, rather than otherwise* **" , was the equivalent, in physics and theology, of the law of contradiction in mathematics:**

" *A proposition cannot be true and false at the same time.* **"**

From this principle he derives the *Principle of the Identity of Indiscernibles* **:**

"There is not a thing as two individuals indiscernible from each other. **"**

By sufficient reason, things cannot be both different and the same at together . If they are different there is a reason why they are different.

The principle is used to argue against the possibility of a vacuum and against Atomism. All parts of a vacuum are indistinguishable. Being indistinguishable they cannot have independent identity. By sufficient reason, no entity can be both One and Many. Therefore the vacuum, or empty space, and even space itself, do not exist. What one calls the 'distance' between two distinguished material objects is only a numerical parameter expressing a certain relationship between these objects.

The non-existence of Space in the absence of Matter is a feature of all cosmological solutions of the Einstein field equations with the exception of de Sitter space.

And in all models of GR the Absolute Space of Newton is abolished in favor of Relative Space. [2]

One also finds many of the essential features of Leibniz's monadology in the calculation of the coefficients of the space-time metric. It is interesting to note that Euler aroused a storm of controversy when, in 1740 , he published a paper at his own expense in which he maintained that Leibniz's Monadology was totally irrelevant to physics and therefore scientifically useless. The controversy was typical of its times: most of it centered around Euler's audacity in publishing his ideas without seeking prior permission from the cranky Leibnizians who clogged the Prussian Academy of Sciences. Probably no one cared one way or the other about its content.

To calculate the exact value of the 10 gravitational metric coefficients at a given point in space and moment in time one has to know the exact location, mass and velocity of every other massive particle in the universe for that moment. One finds a total reflection of the Macrocosm in the Microcosm which (putting aside Leibniz's postulation of the existence of a vital soul in each monad) is the essence of the Monadology.

In theory **, from the dependence of the values of the gravitational metric coefficients on the distribution of matter in the universe it ought to be possible to compute the shape, size and behavior of everything everywhere from the fluctuations of these coefficients at, say, the summit of the Eiffel Tower.**

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[2]Even this is not entirely true anymore: the microwave background radiation, the echo left over from the Big Bang, returns a facsimile of the Ether concept to modern cosmology.

In practice , since no information can travel across space at a speed faster than the speed of light, the time and effort required to gather up all this information would, even theoretically, be infinite: a paradox intrinsic to General Relativity that has delighted humankind for almost a century.

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Conical Gravity

The general equation for the circular conical surface in Cartesian coordinates (x,y,z) in Euclidean 3- space is given by

$$\boxed{z^2 = k^2(x^2 + y^2)}\ .$$

3 "linear" curves are associated with the conical surface :

(1) *The generator lines* passing through the origin.

(2) *The geodesics* . Depending on the magnitude of k, 1 or more geodesics may pass between points on a conic surface not on a generator line. The generators themselves are geodesics.

(3) *The conic sections* obtained through the intersections of the conic surface with planes.

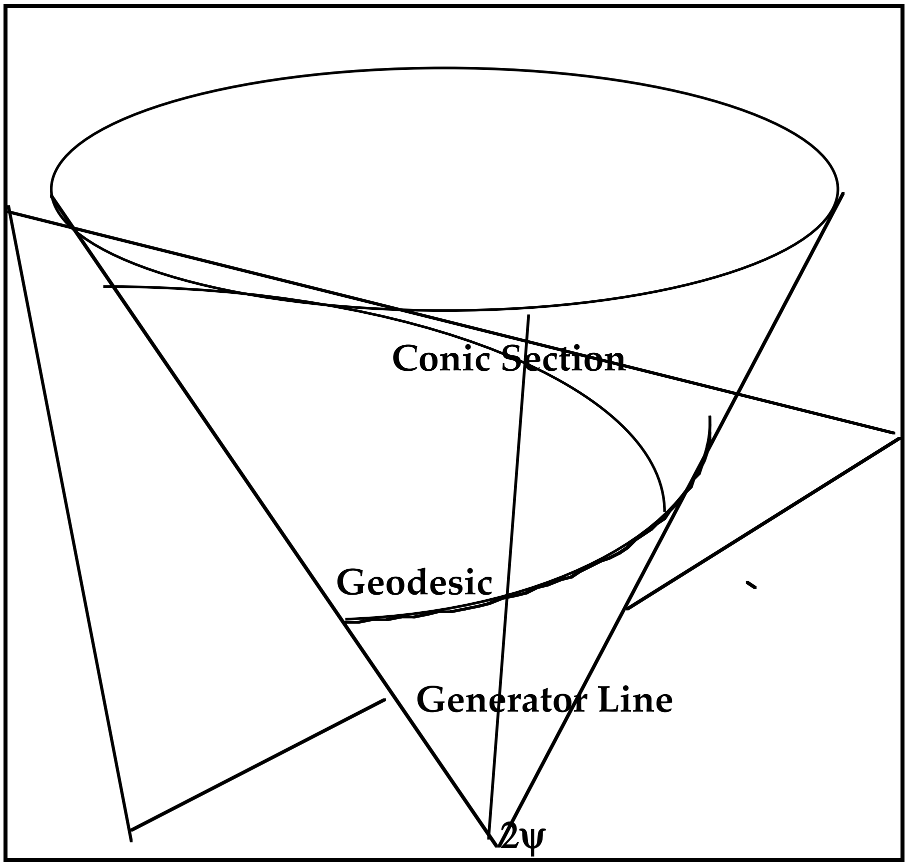


Figure 1

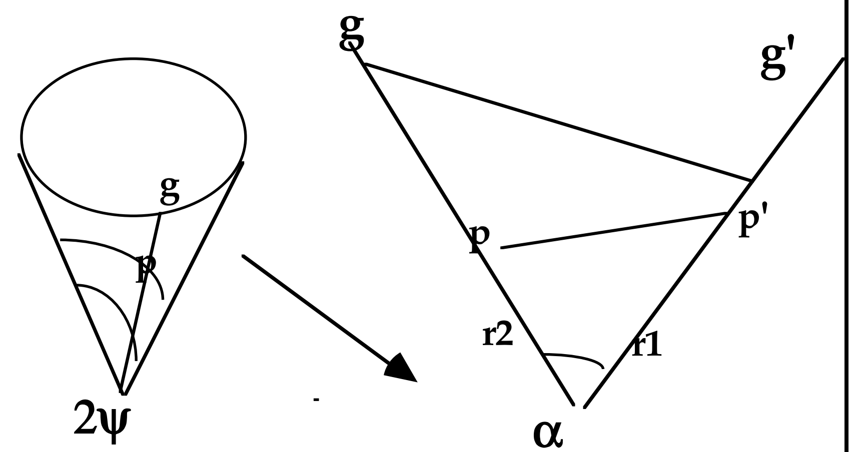
A *generator line* is completely specified by a single point lying on the conic surface:

A *conical geodesic* is specified by two points on the cone. However, here one must be careful. The number of geodesics passing through two points will always be finite, yet it can be any number depending on the positions of the points and the central angle (obtained by cutting it with a plane holding the z-axis) at the vertex of the cone .

Theorem 1

There is a number N = N(k) , which is a function of the central angle at the vertex of the cone, giving the maximum number of conical geodesics passing through any two points.

Conical Geometry provides many examples of simple yet interesting extensions of ordinary plane geometry. Since the Gaussian Curvature at each point is 0, the local geometry on the cone surface is everywhere indistinguishable from Euclidean geometry. We ourselves may well be living on a 4-dimensional cone's surface and not know it until a light beam returns to us from an unusual direction. The geometry of a 3-D cone's surface can easily be visualized by unfolding the cone and laying it flat on the Euclidean plane.



Particle p on a conic surface, on generator line g becomes the pair pp' when opened along g and laid flat.

Figure 2.

The geodesics become ordinary straight lines, while the generators translate into the pencil of lines emanating from the vertex. The conic sections have more complicated equations.

Let the central angle of the cone in 3-space be designated 2ψ , and the central angle of the unfolded cone α ; the reason for the coefficient

2 will become apparent in a moment. In Figure 2 the cone has been unfolded along the generator line g . Let pp' be a line drawn across the flattened sector, intersecting the two copies of the generator line at distances r_1 and r_2 . If $r_1 > r_2$ then it is clear that if the cone be refolded to its original configuration in 3-space, these two points will not coincide . Therefore the generator line gwill intersect the geodesic pp' in two places . Since generators are also geodesics, this already shows that ***every cone whose vertex angle in the corresponding flattened sector is less than π has points between which there are more than one geodesic .***

We now compute the relationship between angles 2ψ and α :

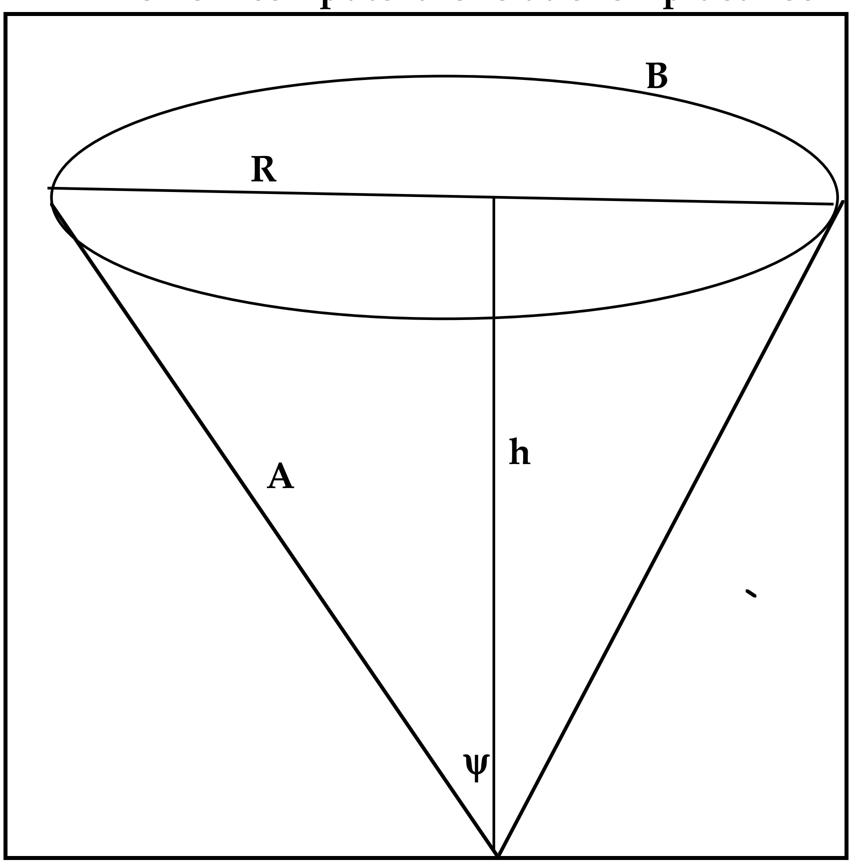


Figure 3

From Figure 3 one sees that if B is the circumference of the upper circle , A the distance from the vertex, then B will be *both* the circumference of a circle of radius R in space, and also the *length* of a circular *arc* of length A in the plane when flattened out. Clearly:

$$\boxed{\begin{array}{l} \sin\psi = \dfrac{R}{A}; \\ B = 2\pi R; \end{array}}$$

$$\boxed{\alpha = \frac{B}{A} = \frac{2\pi R}{A} = 2\pi \sin\psi}$$

Formula I

Gravitation on a Conical Surface.

If the vertex angle α of the plane sector is π , then the central angle
2ψ in 3 space is given by $2\sin^{-1}(1/2) = \pi/3 = 60°$. Therefore it follows that if 2ψ is less than 60°, every non-generator geodesic will self-intersect, but that if it is 60° or larger, the geodesics will not be self intersecting. Let C be a cone surface in 3-space with the vertex removed, and with equations:

$$z^2 = k^2(x^2 + y^2)$$
$$x, y, z \neq 0$$

Any two points p , q on the surface will have the same relationship , *as individuals relative to C* ('indiscernibles' in Leibniz's terms) , yet the pair (p,q) will be distinguishable in its properties from other arbitrary pairs (r,s) on the cone.

Leibniz's arguments for a relative rather than absolute space do not take into account the possibilty that it may have global properties, even under the restriction that the local metric properties be Euclidean.

Generalizing to 3 dimensions, let S be the 3-dimensional cone surface in 4-space (with singular vertex removed) given by:

$$\boxed{\begin{array}{l} w^2 = k^2(x^2 + y^2 + z^2) \\ w, x, y, z \neq 0 \end{array}}$$

Without invoking General Relativity it would possible for us to be living on such a surface without being aware of it. The self-intersection of a geodesic (such as a light ray) would be the only way we could know that space was not Euclidean, and if we are far enough away from the vertex, it's not inconceivable that none of the light rays sent out in the short period of mankind's existence on earth has been exactly on the track that self-intersects on earth, or made its complete circuit.

There are however indirect consequences of living on such a surface owing to a phenomenon we have dubbed *self- gravity* . Let's return to the surface of C and the 2-dimensional model. Imagine that Newton's inverse square law for the gravitational field acts between any two material bodies on C. (This is consistent with the behavior of a central force in 3 -space, since any object moving under the action of a central force and none other will move in a plane.)

Theorem 2

If the central angle 2ψ of a cone surface C on which Newton's law of gravitation operates, is less than 60° then material particles *will move to the vertex under the action of their own gravitational field upon themselves* .

Proof : By formula I, when C is opened up along a generator g and laid flat, it becomes a sector of the Euclidean plane with a vertex angle less than π .

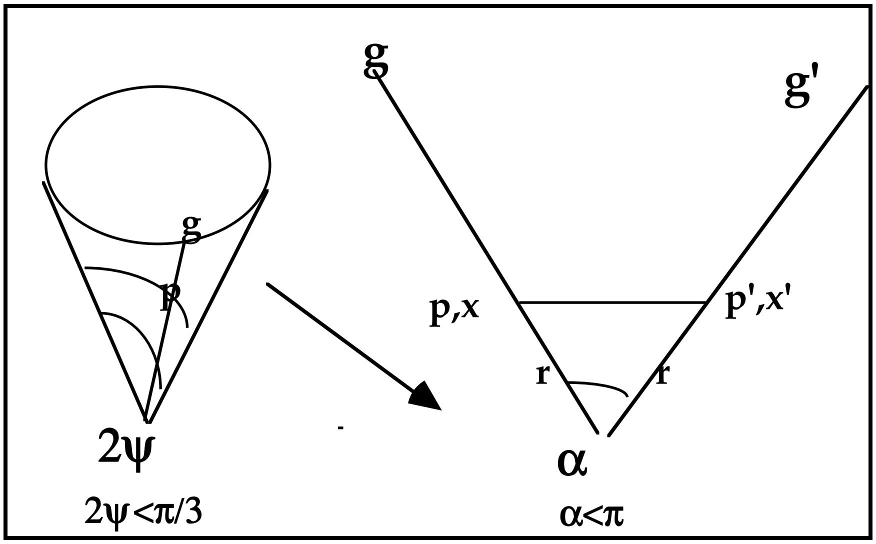


Figure 4

As one sees from Figure 4, when the cone is opened up, the generator line g becomes two lines g and g', while the point x becomes two points x and x' . These are actually the same point when folded back up, while the horizontal line going from x to x' becomes a self-intersecting geodesic. [3]

We place a massive particle p at the point x. The gravitational field that it generates lies entirely on the surface of C. There will be a force vector l from p to p' , and another equal and opposite force vector h from p' to p (*Figure 5*). Observe how each of these make equal and opposite angles with the generator lines g and g' . An easy calculation shows that this angle is given by $\boxed{\beta = \frac{\pi - \alpha}{2}}$.

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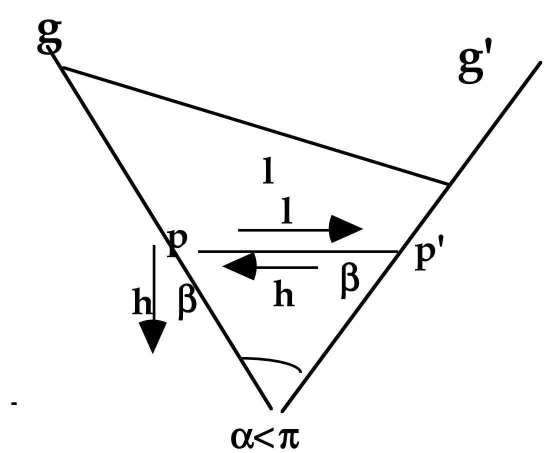
[3]Important Note: this is *not* the same as the circle obtained by cutting C with a plane perpendicular to the axis. If the experiment is made one sees that the geodesic falls below the circle .

To see what needs to be done to calculate the total effect of the self-gravitational force of the particle p on itself, we parallel transport the vector h of p' on the right side over to the left and extend it from p. The resultant looks like this:



The vector h transposed to the p on the left, pointing to the vertex and maintaining a constant angle β with the generators.

Figure 5

Let the force on the left be F_l, that on the right F_h. Assuming a Newtonian inverse square law, these act along the line connecting p with p'. ***Although the force vectors themselves act along the self-intersecting geodesic, the resultant of forces points downwards to the vertex.*** **Let the distance of particle p at time t from the vertex of the cone be given by the function x(t) . Then the distance q along the geodesic between p and p' is clearly** $\boxed{q = 2x\cos\beta = 2x\sin \alpha/2}$.

Newton's law of gravitational attraction gives:

$$\left|F_l\right| = \left|F_r\right| = -M\frac{d^2(q)}{dt^2} = \frac{\gamma(M)^2}{q^2}$$
$$= \frac{\gamma(M)^2}{(2x\cos\beta)^2}$$
$$\frac{d^2(2x\cos\beta)}{dt^2} = -\frac{\gamma M}{(2x\cos\beta)^2};$$
$$\frac{d^2(x)}{dt^2}\Big|_{p-left} = -\frac{\gamma M}{8x^2\cos^3\beta} = -\frac{d^2(x)}{dt^2}\Big|_{p-right}$$

The resultant of forces F_l and F_r add vectorially to give the total self-gravity F_p, and can be calculated from the diagram:

$$F_p = -(\left|F_l\right| + \left|F_r\right|)\cos\beta$$
$$= -\frac{\gamma(M)^2}{4x^2\cos^2\beta} = M\frac{d^2(x)}{dt^2}\Big|_{total}$$

The equation of motion of p under its own self-gravity is gotten by integrating this equation. Let's say that at time t = 0, particle p of mass M is at rest at a distance r = x(0) from the vertex. The first integral of the above equation gives :

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 = E + \frac{\gamma M}{4x\cos^2\beta}$$

E is a constant to be determined by initial conditions. When x =r, dx/dt = 0, and

$$\boxed{E = -\frac{\gamma M}{4r\cos^2\beta}}.$$

Let

$$\boxed{s = \frac{M\gamma}{4\cos^2\beta}}.$$

Then

$$\boxed{\begin{aligned} &\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{s}{x} - \frac{s}{r}; \\ &\frac{dx}{dt} = -\sqrt{2s/r}\left(\frac{r-x}{x}\right) \end{aligned}}$$

The minus sign is used because the velocity vector points to the vertex. The above may also be written as:

$$\boxed{\sqrt{\frac{x}{r-x}}dx = -\sqrt{2s/r}\,dt}$$

This gives the following integral to evaluate:

$$\boxed{\int_x^r \sqrt{\frac{u}{r-u}}du = -(\sqrt{2s/r})t}.$$

By substitution:

$$\boxed{\begin{aligned}&w^2 = \frac{u}{r-u};\\ &u = \frac{w^2}{1+w^2};\\ &du = \frac{2wdw}{(1+w^2)^2};\\ &\sqrt{\frac{u}{r-u}}du = \frac{2w^2dw}{(1+w^2)^2}\end{aligned}}$$

Make the further substitution:

$$\boxed{\begin{aligned}&w = \tan\theta;\\ &2\int\frac{w^2dw}{(1+w^2)^2} = 2\int\frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta d\theta\\ &= 2\int\sin^2\theta d\theta = \theta - \frac{\sin 2\theta}{4}\end{aligned}}$$

In terms of w this becomes : $\boxed{\int \quad = \arctan w - \frac{w}{2(1+w^2)}}$

Substituting u, and restoring the limits on the original integral, we have:

$$\boxed{\begin{aligned}&\left(\sqrt{\frac{2s}{r}}\right)t = \arctan\sqrt{\frac{u}{r-u}} - \frac{(r-u)\sqrt{\frac{u}{r-u}}}{2r}\Bigg|_x^r\\ &= \frac{\pi}{2} - \arctan\left(\sqrt{\frac{x}{r-x}}\right) + \frac{\sqrt{x(r-x)}}{r})\end{aligned}}$$

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Problem : **How long does it take for the particle to reach the vertex?**

Solution **: Set x = 0 in the above equation. Then only the constant term remains:**

$$\boxed{\begin{array}{l} Time = \dfrac{\pi}{2}\dfrac{1}{\sqrt{\dfrac{2s}{r}}}; s = \dfrac{M\gamma}{4\cos^2\beta} \\ \therefore Time = \dfrac{\pi}{2}\dfrac{1}{\sqrt{\dfrac{2\dfrac{\gamma M}{4\cos^2\beta}}{r}}} = \dfrac{\pi}{2}\sqrt{\dfrac{2r\cos^2\beta}{\gamma M}} = \pi\cos\beta(2\gamma)^{-1/2}\sqrt{\dfrac{r}{M}} \end{array}}$$

Hence the time of free fall towards the singularity varies as the square root of the distance and inversely as the square root of the mass. This can also be deduced from Dimensional Analysis, as one does for the period of the moving pendulum.

Problems :

(1) What is the smallest *vertex angle* for which there can exist *two* self-intersecting geodesics through every point on the cone surface?

(2) What is the corresponding *central angle* ?

(3) Derive the complete self-gravity for a particle p of mass M at distance r from the vertex when there are two self-intersecting geodesics.

Solutions **: If the vertex angle (of the flattened sector) is π /3 , then one can lay down successive copies of the sector formed by the**

flattened cone around the origin to form the rays emanating from the origin of a regular hexagon to the vertices:

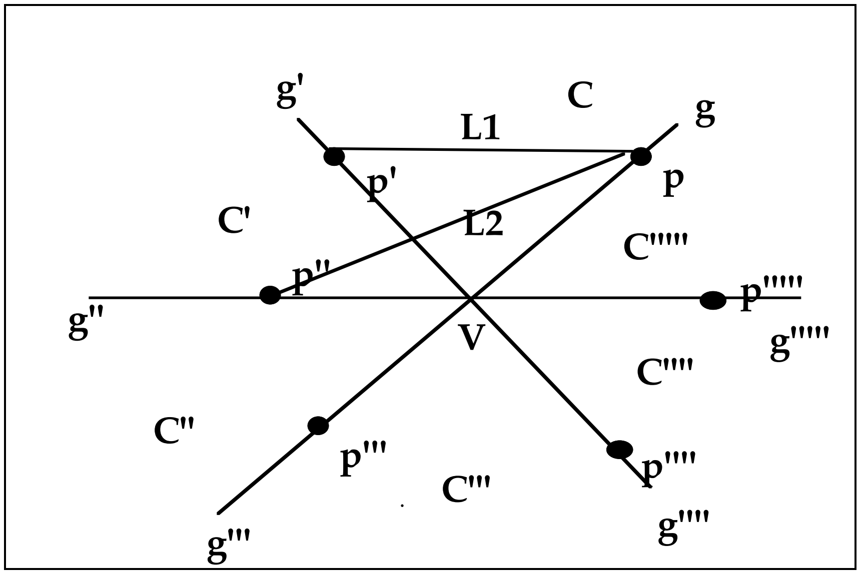

Figure 6

C, C' and so on, are copies of the flattened cone that have been laid in counter-clockwise sequence on the plane. Since the vertex angle is $\pi/3$ one can unfold the cone to make 6 sectors. A straight line drawn between *any* two points on the diagram corresponds to a geodesic on the cone. The points p , p' ....p''''' , are images of the same point, even as the lines g. g' ....g''''' are images of the same generator line.

The line L_1 between p and p' is a simple loop of a geodesic beginning and ending in p. The line L_2 is also a geodesic, wrapping once about the cone to re-intersect with p before moving on.

Clearly, lines between p' and p'', p'' and p''' , etc., are images of the same loop L_1 . Likewise a line between p' and p''' is the same as L_2 . Note how the line between p and p''' is just a generator line going to the vertex and returning. The vertex is a singularity, so pp''' is not a geodesic.

By inspection of the above diagram, one may write down the self-gravity equation which pulls the particle p down to the vertex. The loop L_1 is identical to the loop in the previous problem. We need therefore only
write down the additional self-attraction equation of L_2, form the resultant of forces, and combine the two equations for L_1 and L_2.

We compute the length of the loop L_2, and the angle that it makes with the generator g .

Let the distance of p from the vertex of the cone be given by the variable x. Then the distances from p to p' is given by

$$\boxed{h_1 = 2x\sin \pi/6 = x;}$$

while that from p to p'' is given by:

$$\boxed{h_2 = 2x\sin 2\pi/6 = x\sqrt{3}}.$$

The law of Newtonian attraction is

$$\boxed{\begin{array}{l} |F_l| = |F_r| = -M\dfrac{d^2(h)}{dt^2} = \dfrac{\gamma(M)^2}{h^2}; \\ \dfrac{d^2(2x\cos\beta)}{dt^2} = -\dfrac{\gamma M}{(2x\cos\beta)^2}; \\ \dfrac{d^2(x)}{dt^2}\Big|_{p-left} = -\dfrac{\gamma M}{8x^2\cos^3\beta} = -\dfrac{d^2(x)}{dt^2}\Big|_{p-right} \end{array}}$$

The resultant of forces F_l and F_r is:

$$\boxed{\begin{aligned} &\frac{F_{(p,p')}}{M} = \frac{\gamma M}{4x^2\, 1/4} = \frac{\gamma M}{x^2} \\ &\frac{F_{(p',p'')}}{M} = \frac{\gamma M}{4x^2\, 3/4} = \frac{\gamma M}{3x^2} \\ &\therefore d^2x/dt^2 = 4/3\left(\frac{\gamma M}{x^2}\right) \end{aligned}}$$

This is equivalent to an attraction of a mass of 4/3 the mass of the particle p, at a distance x from p.

In general, if we make the angle α smaller than π /3 but larger than π/4 , the total attraction is given by the resultant force between p and p', and that between p and p" :

$$\boxed{\begin{aligned} &(1)\frac{F_{(p,p')}}{M} = \frac{\gamma M}{4x^2\, 1/4} = \frac{\gamma M}{x^2} \\ &(2)\frac{F_{(p',p'')}}{M} = \frac{\gamma M}{4x^2\, 3/4} = \frac{\gamma M}{3x^2} \\ &\therefore d^2x/dt^2 = \frac{\gamma M}{4x^2 \sin^2 \alpha/2} + \frac{\gamma M}{4x^2 \sin^2 \alpha} \\ &= \left(\frac{1}{\sin^2 \alpha/2} + \frac{1}{4\sin^2 \alpha/2 \cos^2 \alpha/2}\right)\frac{\gamma M}{4x^2} = \left(\frac{4\cos^2 \alpha/2 + 1}{4\sin^2 \alpha/2 \cos^2 \alpha/2}\right)\frac{\gamma M}{4x^2} \end{aligned}}$$

In the range that has been specified, the *effective mass* of the attraction is lies between 4/3 and 5/4 of the mass M of the particle p.

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