Conical Gravity

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July,2004

The general equation for the circular conical surface in Cartesian coordinates (x,y,z) in Euclidean 3- space is given by

$$\boxed{z^2 = k^2(x^2 + y^2)} \ .$$

3 "linear" curves are associated with the conical surface :

(1) *The generator lines* **passing through the origin.**

(2) *The geodesics* **. Depending on the magnitude of k, 1 or more geodesics may pass between points on a conic surface not on a generator line. The generators themselves are geodesics.**

(3) *The conic sections* **obtained through the intersections of the conic surface with planes.**

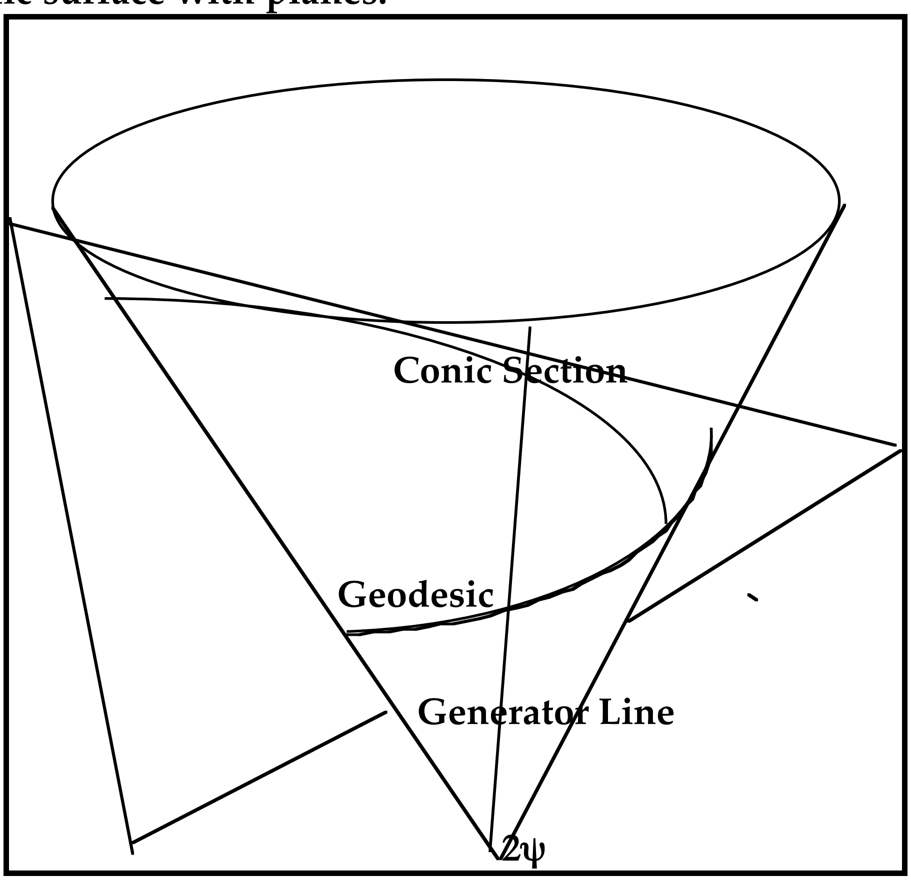


Figure 1

A *generator line* is completely specified by a single point lying on the conic surface:

A *conical geodesic* is specified by two points on the cone. However, here one must be careful. The number of geodesics passing through two points will always be finite, yet it can be any number depending on the positions of the points and the central angle (obtained by cutting it with a plane holding the z-axis) at the vertex of the cone .

Theorem 1

There is a number N = N(k) , which is a function of the central angle at the vertex of the cone, giving the maximum number of conical geodesics passing through any two points.

Conical Geometry provides many examples of simple yet interesting extensions of ordinary plane geometry. Since the Gaussian Curvature at each point is 0, the local geometry on the cone surface is everywhere indistinguishable from Euclidean geometry. We ourselves may well be living on a 4-dimensional cone's surface and not know it until a light beam returns to us from an unusual direction. The geometry of a 3-D cone's surface can easily be visualized by unfolding the cone and laying it flat on the Euclidean plane.

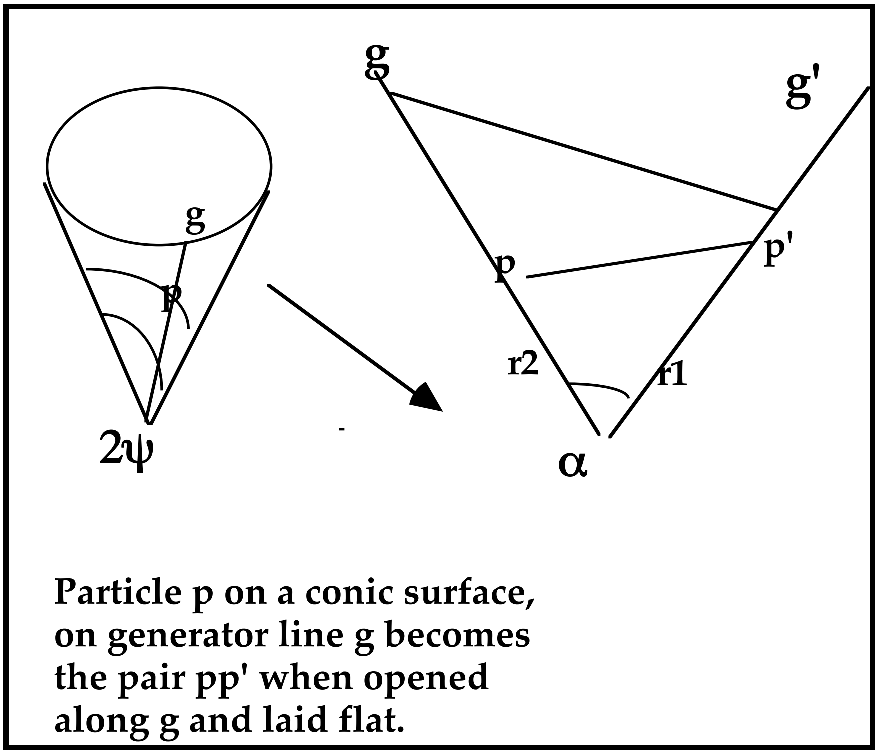


Figure 2

The geodesics become ordinary straight lines, while the generators translate into the pencil of lines emanating from the vertex. The conic sections have more complicated equations.

Let the central angle of the cone in 3-space be designated 2ψ , and the central angle of the unfolded cone α ; the reason for the coefficient 2 will become apparent in a moment. In Figure 2 the cone has been unfolded along the generator line g . Let pp' be a line drawn across the flattened sector, intersecting the two copies of the generator line at distances r_1 and r_2 . If $r_1 > r_2$ then it is clear that if the cone be refolded to its original configuration in 3-space, these two points will not coincide . Therefore the generator line g will intersect the geodesic pp' in two places . Since generators are also geodesics, this already shows that *every cone* *whose vertex angle in the corresponding flattened sector is less than π has points between which there are more than one geodesic .*

We now compute the relationship between angles 2ψ and α :

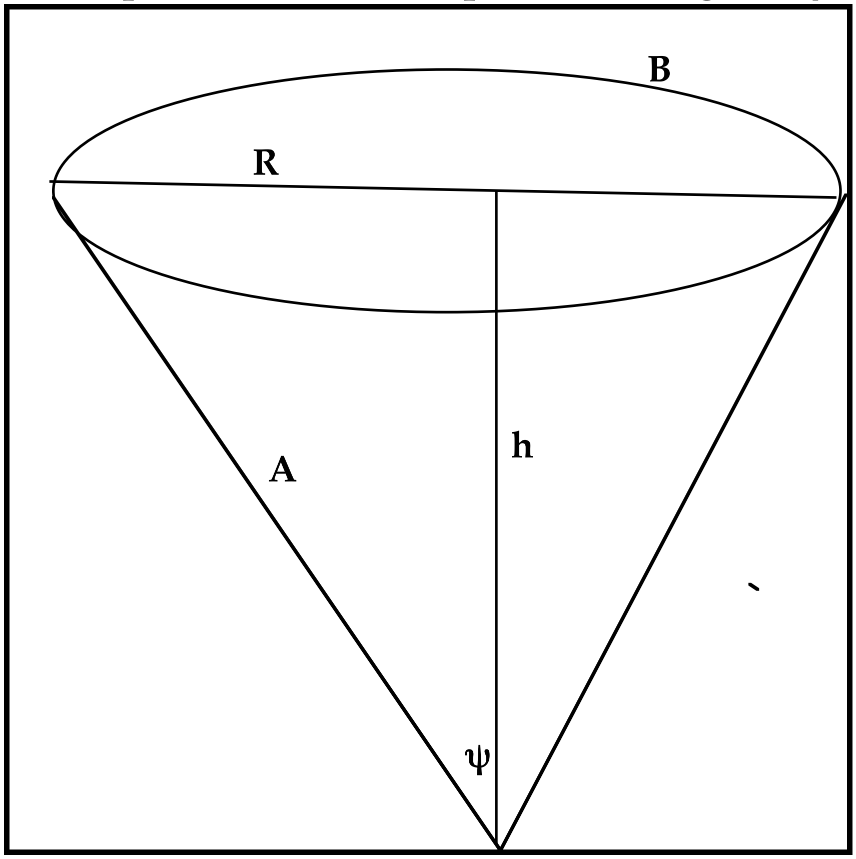


Figure 3

From Figure 3 one sees that if B is the circumference of the upper circle , A the distance from the vertex, then B will be *both* the circumference of a circle of radius R in space, and also the *length* of a circular *arc* of length A in the plane when flattened out. Clearly:

$$\boxed{\begin{array}{l}\sin\psi = \dfrac{R}{A};\\ B = 2\pi R;\end{array}}$$

$$\boxed{\alpha = \frac{B}{A} = \frac{2\pi R}{A} = 2\pi \sin\psi}$$

Formula I

Gravitation on a 2-Cone Surface

If the vertex angle α of the plane sector is π , then the central angle 2ψ in 3 space is given by $2\sin^{-1}(1/2) = \pi/3 = 60°$. Therefore it follows that if 2ψ is less than 60°, every non-generator geodesic will self-intersect, but that if it is 60° or larger, the geodesics will not be self intersecting. Let C be a cone surface in 3-space with the vertex removed, and with equations:

$$z^2 = k^2(x^2 + y^2)$$
$$x, y, z \neq 0$$

Any two points p , q on the surface will have the same relationship , ***as individuals relative to C*** **('indiscernibles' in Leibniz's terms) , yet the pair (p,q) will be distinguishable in its properties from other arbitrary pairs (r,s) on the cone.**

Leibniz's arguments for a relative rather than absolute space do not take into account the possibility that it may have global properties, even under the restriction that the local metric properties be Euclidean.

Generalizing to 3 dimensions, let S be the 3-dimensional cone surface in 4-space (with singular vertex removed) given by:

$$\boxed{\begin{array}{l} w^2 = k^2(x^2 + y^2 + z^2) \\ w, x, y, z \neq 0 \end{array}}$$

Without invoking General Relativity it would possible for us to be living on such a surface without being aware of it. The

self-intersection of a geodesic (such as a light ray) would be the only way we could know that space was not Euclidean, and if we are far enough away from the vertex, it's not inconceivable that none of the light rays sent out in the short period of mankind's existence on earth has been exactly on the track that self-intersects on earth, or made its complete circuit.

There are however indirect consequences of living on such a surface owing to a phenomenon we have dubbed *self- gravity* **. Let's return to the surface of C and the 2-dimensional model. Imagine that Newton's inverse square law for the gravitational field acts between any two material bodies on C. (This is consistent with the behavior of a central force in 3 -space, since any object moving under the action of a central force and none other will move in a plane.)**

Theorem 2

If the central angle 2ψ of a cone surface C on which Newton's law of gravitation operates, is less than 60° then material particles *will move to the vertex under the action of their own gravitational field upon themselves* **.**

Proof **: By formula I, when C is opened up along a generator g and laid flat, it becomes a sector of the Euclidean plane with a vertex angle less than π .**

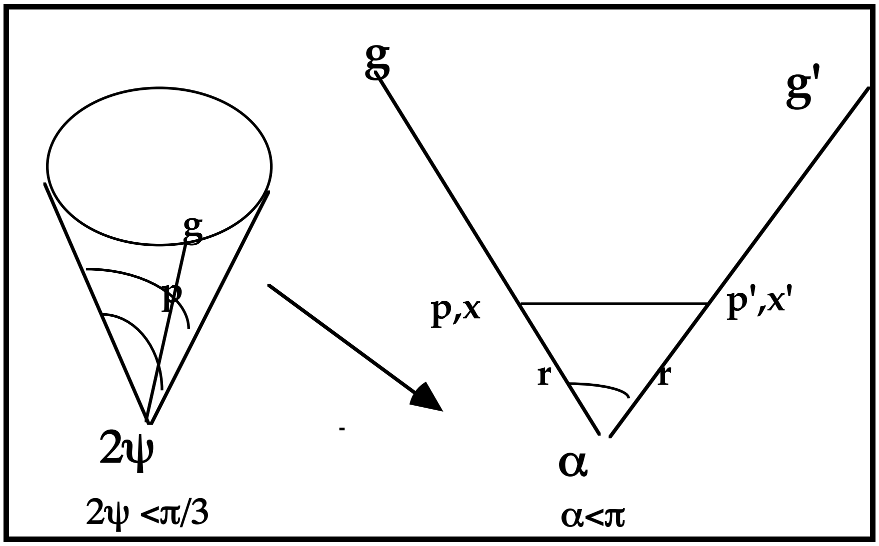


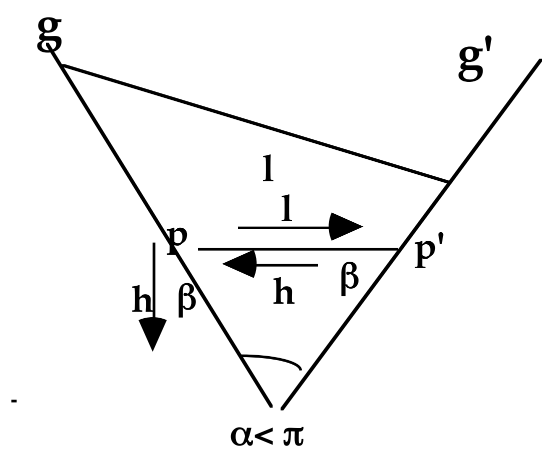
Figure 4

As one sees from Figure 4, when the cone is opened up, the generator line g becomes two lines g and g', while the point x becomes two points x and x' . These are actually the same point when folded back up, while the horizontal line going from x to x' becomes a self-intersecting geodesic. [1] We place a massive particle p at the point x. The gravitational field that it generates lies entirely on the surface of C. There will be a force vector l from p to p' , and another equal and opposite force vector h from p' to p (*Figure 5*). Observe how each of these make equal and opposite angles with the generator lines g and g' . An easy calculation shows that this angle is given by

$$\boxed{\beta = \frac{\pi - \alpha}{2}}.$$

[1]**Important Note: this is *not* the same as the circle obtained by cutting C with a plane perpendicular to the axis. If the experiment is made one sees that the geodesic falls below the circle .**

To see what needs to be done to calculate the total effect of the self-gravitational force of the particle p on itself, we parallel transport the vector h of p' on the right side over to the left and extend it from p. The resultant looks like this:



The vector h transposed to the p on the left, pointing to the vertex and maintaining a constant angle β with the generators.

Figure 5

Let the force on the left be F_l, that on the right F_h. Assuming a Newtonian inverse square law, these act along the line connecting p with p'. *Although the force vectors themselves act along the self-intersecting geodesic, the resultant of forces points downwards to the vertex.* Let the distance of particle p at time t from the vertex of the cone be given by the function x(t) . Then the distance q along the geodesic between p and p' is clearly:

$$\boxed{q = 2x\cos\beta = 2x\sin \alpha/2}.$$

Newton's law of gravitational attraction gives:

$$\left|F_l\right| = \left|F_r\right| = -M\frac{d^2(q)}{dt^2} = \frac{\gamma(M)^2}{q^2}$$
$$= \frac{\gamma(M)^2}{(2x\cos\beta)^2}$$
$$\frac{d^2(2x\cos\beta)}{dt^2} = -\frac{\gamma M}{(2x\cos\beta)^2};$$
$$\frac{d^2(x)}{dt^2}\Big|_{p-left} = -\frac{\gamma M}{8x^2\cos^3\beta} = -\frac{d^2(x)}{dt^2}\Big|_{p-right}$$

The resultant of forces F_l and F_r add vectorialy to give the total self-gravity F_p, and can be calculated from the diagram:

$$F_p = -(\left|F_l\right| + \left|F_r\right|)\cos\beta$$
$$= -\frac{\gamma(M)^2}{4x^2\cos^2\beta} = M\frac{d^2(x)}{dt^2}\Big|_{total}$$

The equation of motion of p under its own self-gravity is gotten by integrating this equation. Let's say that at time t = 0, particle p of mass M is at rest at a distance r = x(0) from the vertex. The first integral of the above equation gives :

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 = E + \frac{\gamma M}{4x\cos^2\beta}$$

E is a constant to be determined by initial conditions. When x =r, dx/dt = 0, and

$$\boxed{E = -\frac{\gamma M}{4r\cos^2\beta}}.$$

Let

$$\boxed{s = \frac{M\gamma}{4\cos^2\beta}}.$$

Then

$$\boxed{\begin{array}{l} \frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{s}{x} - \frac{s}{r}; \\ \frac{dx}{dt} = -\sqrt{2s/r}\left(\frac{r-x}{x}\right) \end{array}}$$

The minus sign is used because the velocity vector points to the vertex. The above may also be written as:

$$\boxed{\sqrt{\frac{x}{r-x}}dx = -\sqrt{2s/r}\,dt}$$

This gives the following integral to evaluate:

$$\boxed{\int_x^r \sqrt{\frac{u}{r-u}}du = -(\sqrt{2s/r})t}.$$

By substitution:

$$\boxed{\begin{aligned} & w^2 = \frac{u}{r-u}; \\ & u = \frac{w^2}{1+w^2}; \\ & du = \frac{2wdw}{(1+w^2)^2}; \\ & \sqrt{\frac{u}{r-u}}du = \frac{2w^2dw}{(1+w^2)^2} \end{aligned}}$$

Make the further substitution:

$$\boxed{\begin{aligned} & w = \tan\theta; \\ & 2\int\frac{w^2dw}{(1+w^2)^2} = 2\int\frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta d\theta \\ & = 2\int\sin^2\theta d\theta = \theta - \frac{\sin 2\theta}{4} \end{aligned}}$$

In terms of w this becomes :

$$\boxed{\int \quad = \arctan w - \frac{w}{2(1+w^2)}}$$

Substituting u, and restoring the limits on the original integral:

$$\boxed{\begin{aligned} & \left(\sqrt{\frac{2s}{r}}\right)t = \arctan\sqrt{\frac{u}{r-u}} - \left.\frac{(r-u)\sqrt{\frac{u}{r-u}}}{2r}\right|_x^r \\ & = \frac{\pi}{2} - \arctan\left(\sqrt{\frac{x}{r-x}}\right) + \frac{\sqrt{x(r-x)}}{r}) \end{aligned}}$$

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***Problem* : How long does it take for the particle to reach the vertex?**

***Solution* : Set x = 0 in the above equation. Then only the constant term remains:**

$$Time = \frac{\pi}{2}\frac{1}{\sqrt{\frac{2s}{r}}}; s = \frac{M\gamma}{4\cos^2\beta}$$

$$\therefore Time = \frac{\pi}{2}\frac{1}{\sqrt{\frac{2\frac{\gamma M}{4\cos^2\beta}}{r}}} = \frac{\pi}{2}\sqrt{\frac{2r\cos^2\beta}{\gamma M}} = \pi\cos\beta(2\gamma)^{-1/2}\sqrt{\frac{r}{M}}$$

Hence the time of free fall towards the singularity varies as the square root of the distance and inversely as the square root of the mass. This can also be deduced from Dimensional Analysis, as one does for the period of the moving pendulum.

Problems :

(1) What is the smallest *vertex angle* for which there can exist *two* self-intersecting geodesics through every point on the cone surface?

(2) What is the corresponding *central angle* ?

(3) Derive the complete self-gravity for a particle p of mass M at distance r from the vertex when there are two self-intersecting geodesics.

***Solutions* : If the vertex angle (of the flattened sector) is π /3 , then one can lay down successive copies of the sector formed**

by the flattened cone around the origin to form the rays emanating from the origin of a regular hexagon to the vertices:

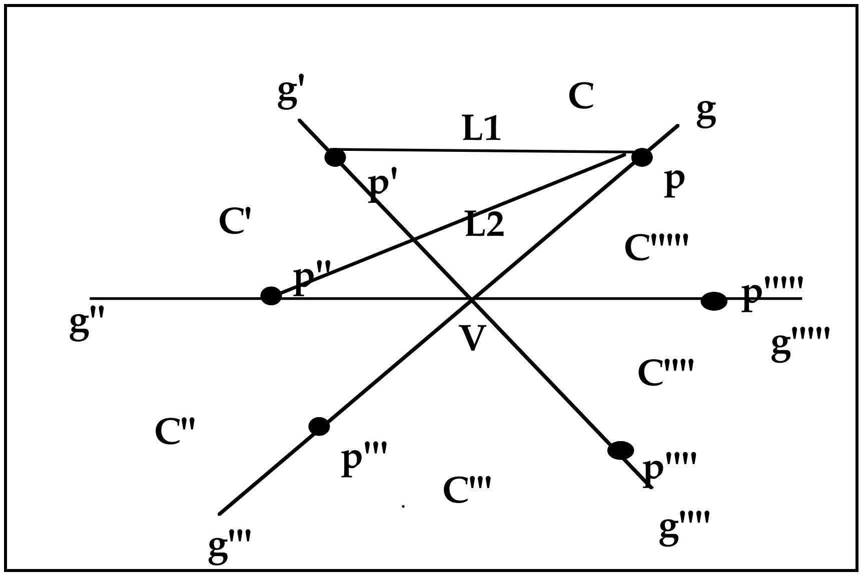

Figure 6

C, C' and so on, are copies of the flattened cone that have been laid in counter-clockwise sequence on the plane. Since the vertex angle is π/3 one can unfold the cone to make 6 sectors. A straight line drawn between *any* two points on the diagram corresponds to a geodesic on the cone. The points p , p' ....p''''' , are images of the same point, even as the lines g. g' ....g''''' are images of the same generator line.

The line L_1 between p and p' is a simple loop of a geodesic beginning and ending in p. The line L_2 is also a geodesic, wrapping once about the cone to re-intersect with p before moving on.

Clearly, lines between p' and p'', p'' and p''' , etc., are images of the same loop L_1 . Likewise a line between p' and p''' is the same as L_2 . Note how the line between p and p''' is just a

generator line going to the vertex and returning. The vertex is a singularity, so pp''' is not a geodesic.

By inspection of the above diagram, one may write down the self-gravity equation which pulls the particle p down to the vertex. The loop L_1 is identical to the loop in the previous problem. We need therefore only write down the additional self-attraction equation of L_2, form the resultant of forces, and combine the two equations for L_1 and L_2 .

We compute the length of the loop L_2, and the angle that it makes with the generator g .

Let the distance of p from the vertex of the cone be given by the variable x. Then the distances from p to p' is given by

$$\boxed{h_1 = 2x\sin \pi/6 = x;}$$

while that from p to p'' is given by:

$$\boxed{h_2 = 2x\sin 2\pi/6 = x\sqrt{3}}.$$

The law of Newtonian attraction is

$$\boxed{\begin{gathered}|F_l| = |F_r| = -M\frac{d^2(h)}{dt^2} = \frac{\gamma(M)^2}{h^2};\\ \frac{d^2(2x\cos\beta)}{dt^2} = -\frac{\gamma M}{(2x\cos\beta)^2};\\ \frac{d^2(x)}{dt^2}\Big|_{p-left} = -\frac{\gamma M}{8x^2\cos^3\beta} = -\frac{d^2(x)}{dt^2}\Big|_{p-right}\end{gathered}}$$

The resultant of forces F_l and F_r is:

$$\boxed{\begin{aligned}&\frac{F_{(p,p')}}{M}=\frac{\gamma M}{4x^2\,1/4}=\frac{\gamma M}{x^2}\\&\frac{F_{(p',p'')}}{M}=\frac{\gamma M}{4x^2\,3/4}=\frac{\gamma M}{3x^2}\\&\therefore d^2x/dt^2=4/3\left(\frac{\gamma M}{x^2}\right)\end{aligned}}$$

This is equivalent to an attraction of a mass of 4/3 the mass of the particle p, at a distance x from p.

In general, if we make the angle α smaller than π /3 but larger than π/4 , the total attraction is given by the resultant force between p and p', and that between p and p" :

$$\boxed{\begin{aligned}&(1)\frac{F_{(p,p')}}{M}=\frac{\gamma M}{4x^2\,1/4}=\frac{\gamma M}{x^2}\\&(2)\frac{F_{(p',p'')}}{M}=\frac{\gamma M}{4x^2\,3/4}=\frac{\gamma M}{3x^2}\\&\therefore d^2x/dt^2=\frac{\gamma M}{4x^2\sin^2\alpha/2}+\frac{\gamma M}{4x^2\sin^2\alpha}\\&=\left(\frac{1}{\sin^2\alpha/2}+\frac{1}{4\sin^2\alpha/2\cos^2\alpha/2}\right)\frac{\gamma M}{4x^2}=\left(\frac{4\cos^2\alpha/2+1}{4\sin^2\alpha/2\cos^2\alpha/2}\right)\frac{\gamma M}{4x^2}\end{aligned}}$$

In the range that has been specified, the *effective mass* of the attraction lies between 4/3 and 5/4 of the mass M of p.

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