Trains and Fly

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There is a length of track of 100 miles between two trains which are moving towards each other at 50 mph. A fly is caught between them. It fly begins from the left at 90 mph. Everything after that is momentum exchange. The tracks are frictionless there is no air resistence. No fuel or locomotion other than momentum exchange in collisions.

The mass of each train is M. The mass of the fly is ε . From sources on the Internet we have derived these values for M and ε :

200 flies = 0.22 grams

1 fly =0.0011 grams ~ 10^{-3} grams

For a train with tender of class CSA-1

Weight = 876,000 lbs. = $\boxed{\dfrac{876,000}{2.21} \times 10^3\, grs.}$

Let q = ε /M .

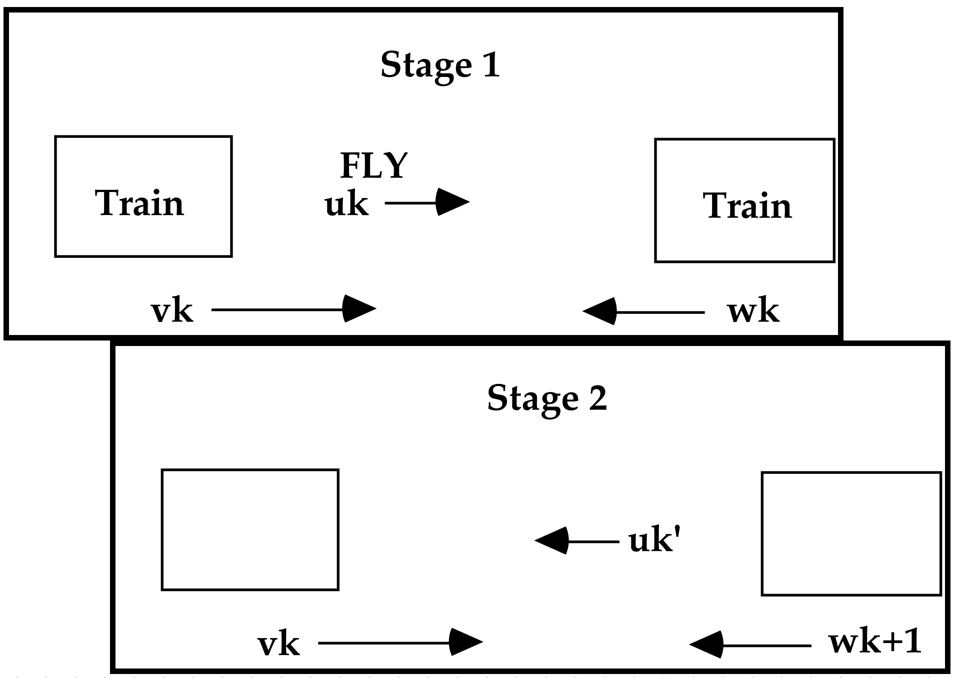
Then

$$\boxed{\begin{aligned} q &= \left(\frac{876,000}{2.21} \times 10^3\right)^{-1} \times 10^{-3} \\ &= \frac{2.21}{876} \times 10^{-9} \approx 0.25 \times 10^{-11} \end{aligned}}$$

That is to say, the train is of the order of 4x100 billion times heavier than the fly. The ratio q will become important as the analysis proceeds.

For this problem (unlike the classical von Neumann mathematics problem) the distances themselves are unimportant. We look instead at energy and momentum exchanges under the restrictions of conservation of energy and conservation of momentum. The fly goes from left to right, collides with the right-hand train, is bounced back with higher momentum and collides with the left train, sending it off again from left to right. This entire process is treated as a single cycle , indexed by k = 1,2,3..... The two stages are clear from the diagrams:

CYCLE k



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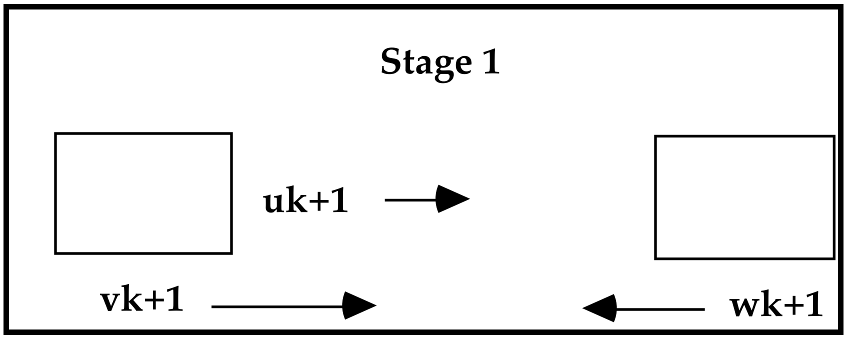
CYCLE k+1



In stage 1 of cycle k , the fly, with velocity u_k , moves from left to right to intersect the train coming towards it with velocity w_k . After their intersection the fly moves from right to left with velocity u_k' , the train on the right has velocity w_{k+1} . When the fly next intersects the train on the left, its velocity will be altered from u_k' to u_{k+1}, and the velocity of the train on the left will be changed to v_{k+1} , initiating the next cycle .

The equations corresponding to a complete cycle are given below:
Let J = Total Momentum at the beginning = 90 ε + 50M-50M = 90ε .
E = Total Energy at the beginning = 1/2 ($(90)^2 \varepsilon + 2(50)^2 M$)
Then:

$$
\begin{array}{l}
(i) M(v_k + w_k) + \varepsilon u_k = J \\
(ii) M(v_k^2 + w_k^2) + \varepsilon u_k^2 = 2E \\
(iii) M(v_k + w_{k+1}) + \varepsilon u_k' = J \\
(iv) M(v_k^2 + w_{k+1}^2) + \varepsilon (u_k')^2 = 2E \\
(v) M(v_{k+1} + w_{k+1}) + \varepsilon u_{k+1} = J \\
(vi) M(v_{k+1}^2 + w_{k+1}^2) + \varepsilon u_{k+1}^2 = 2E
\end{array}
$$

A quick examination shows that we have 6 equations in 7 unknowns. After simplifying this set of equations we will select one variable from which all the rest may be derived.

In the stage one momentum transfer the velocities u_k and w_k are transformed into u_k' and w_{k+1} . It is important to keep in mind that the positive direction is from left to right and that, in the initial stages at least, w_k is a negative number. Classically the momentum transfer is given by:

$$\mu = \frac{2M\varepsilon(w_k - u_k)}{M + \varepsilon}$$

$$(1)\ u_k' = (u_k + \mu/\varepsilon) = (u_k + \frac{2M(w_k - u_k)}{M + \varepsilon}) = \frac{2Mw_k - u_k(M - \varepsilon)}{M + \varepsilon}$$

$$(2)\ w_{k+1} = w_k - \mu/M = \ldots = \frac{2\varepsilon u_k + w_k(M - \varepsilon)}{M + \varepsilon}$$

Since u_k is initially very tiny, and M is huge compared to ε , (by a factor of 10^{11} !) , the collisions of stage 1 result in an approximate net increase of twice the velocity wk to the velocity of the fly, while that of the train is almost negligeable. This will change over time after a great many collisions have taken place. The collision in stage 2 is between the returning fly and the train on the left:

$$\mu' = \frac{2M\varepsilon(v_k - u_k')}{M+\varepsilon}$$

$$(3)\; u_{k+1} = u_k' + \mu'/\varepsilon = u_k' + \frac{2M(v_k - u_k')}{M+\varepsilon}$$

$$= \frac{2Mv_k - u_k'(M-\varepsilon)}{M+\varepsilon} = \frac{2Mv_k - (M-\varepsilon)[\frac{2Mw_k - (M-\varepsilon)u_k}{M+\varepsilon}]}{M+\varepsilon}$$

$$= \ldots = (\frac{2M}{M+\varepsilon})v_k - \frac{2M(M-\varepsilon)}{(M+\varepsilon)^2}w_k + (\frac{M-\varepsilon}{M+\varepsilon})^2 u_k$$

$$(4)\, v_{k+1} = v_k - \mu'/M = v_k - \frac{2\varepsilon}{M+\varepsilon}(v_k - u_k')$$

$$= \ldots = (\frac{M-\varepsilon}{M+\varepsilon})v_k - \frac{2\varepsilon(M-\varepsilon)}{(M+\varepsilon)^2}u_k + \frac{4M\varepsilon}{(M+\varepsilon)^2}w_k$$

It is possible to combine these equations with the expressions for momentum and energy so as to express every variable as a function of u_k .

Rewrite the conservation equations (i) , (ii) , as

$$Mw_k = J - \varepsilon u_k - Mv_k$$

$$(ii)\, M(v_k^2 + (\frac{J - \varepsilon u_k - Mv_k}{M})^2) = 2E - \varepsilon u_k^2$$

Expanding the left side and transposing:

$$
\begin{aligned}
&Mw_k = J - \varepsilon u_k - Mv_k \\
&M(v_k^2) + (\frac{J - \varepsilon u_k - Mv_k}{M})^2) \\
&= 2E - \varepsilon u_k^2 \\
&Mv_k^2 + \tfrac{1}{M}(J^2 + \varepsilon^2 u_k^2 + M^2 v_k^2 - 2J\varepsilon u_k - 2JMv_k + 2\varepsilon M u_k v_k) \\
&= 2E - \varepsilon u_k^2; \\
&\ldots \\
&2M^2 v_k^2 + v_k\{2M\varepsilon u_k - 2MJ\} \\
&+[\{u_k^2 \varepsilon(M + \varepsilon) - 2J\varepsilon u_k\} + \{J^2 - 2ME\}] = 0
\end{aligned}
$$

This is a quadratic in the variable v_k . Its coefficients are polynomials in u_k

$$
\begin{aligned}
&P(x) = \{2M\varepsilon x - 2MJ\} \\
&Q(x) = x^2 \varepsilon(M + \varepsilon) - 2J\varepsilon x + (J^2 - 2ME)
\end{aligned}
$$

As equations (i) and (ii) are symmetrical in v_k and w_k , they appear as the two solutions of the quadratic, and are given by:

$$
(A): v_k, w_k = \frac{-P(u_k) \pm \sqrt{P^2 - 8M^2 Q}}{4M^2}
$$

Since ε(M+ε) is infinitesimal at the beginning of the process, Q(x) will be minus. Therefore, initially v_k is the positive solution, w_k the negative. Finally, u_{k+1} may be expressed as a function of u_k by virtue of equation (3) above. Note once again that u_k is always positive, with u_k' being the velocity in the negative direction.

THEOREM:

The speeds of the trains at every cycle are a function only of the velocity of the fly. Likewise the velocity of the fly in each cycle can be computed as a function of the velocity of the fly in the previous cycle.

For many many thousands of hits we can assume that $\varepsilon = 0$. Then we see that

$$\boxed{\begin{array}{l} P(u_k) \approx 0; Q(u_k) \approx -2ME \approx -5000M^2 \\ v_k \approx 50; w_k \approx -50 \\ u_{k+1} \approx 2v_k - 2w_k + u_k = 200 + u_k; \\ \therefore u_n \approx u_0 + 200n = 200n + 90 \end{array}}$$

In the initial phase, the velocity of the fly increases by 200 mph with each cycle. Eventually the size of this increment will diminish. However, the total velocity of the fly will continue to rise until the trains are arrested and turned back by the fly. Without knowing the details we are able, from the above equations, to estimate the value of u_n at which v_n and w_n are closest to 0. This is the same as saying that $Q(u_n)$ is approximately 0 , or

$$\boxed{\begin{array}{l} Q(u_n) = u_n^2\varepsilon(M+\varepsilon) - 180\varepsilon^2 u_n + (J^2 - 2ME) = 0 \\ J = 90\varepsilon; E = \frac{1}{2}((90)^2\varepsilon + 2(50)^2 M) \end{array}}$$

This is another quadratic, in u_n , with (positive) solution

$$\boxed{u_n = \frac{180\varepsilon^2 + \sqrt{(180\varepsilon^2)^2 + 4\varepsilon(M+\varepsilon)(2ME - J^2)}}{2\varepsilon(M+\varepsilon)}}$$

We can ignore the tiny amount at the left of the right-hand side, and get:

$$\boxed{\begin{array}{l} u_n \approx \dfrac{\sqrt{(180\varepsilon^2)^2 + 4\varepsilon(M+\varepsilon)(2ME - J^2)}}{2\varepsilon(M+\varepsilon)} \\ = \sqrt{\dfrac{(180\varepsilon^2)^2 + 4\varepsilon(M+\varepsilon)(2ME - J^2)}{4\varepsilon^2(M+\varepsilon)^2}} \\ = \sqrt{(\dfrac{90\varepsilon}{M+\varepsilon})^2 - \dfrac{(90)^2\varepsilon}{M+\varepsilon} + \dfrac{2ME}{\varepsilon(M+\varepsilon)}} \end{array}}$$

The first two terms in this expression are negligible. Therefore one estimates u_n as :

$$u_n \approx \sqrt{\frac{2ME}{\varepsilon M}} = \sqrt{\frac{2E}{\varepsilon}} \approx \sqrt{\frac{2500M}{\varepsilon}} = 50\sqrt{\frac{M}{\varepsilon}} = \frac{50}{\sqrt{q}}!$$

It was stated before that q is of the order of 10^{11}. Therefore, the peak speed which the fly must obtain before the process is reversed is about $S \approx 50 \times \sqrt{10} \times 10^5 = 5 \times \sqrt{10} \times 10^6 mph$. As there are 3600 seconds in the hour we can estimate the closeness of S to the speed of light

$$S \approx \frac{3 \times 10^7}{7200} mps = \frac{10^5}{24} mps$$

Qualitative Analysis

PHASE I: The initial phase is that in which the weight (and momentum) of the fly may be considered negligible. Over many thousands of recoils, its velocity will augment by 200 mph. With its increase of speed, the trains are slowed down infinitesimally.

PHASE II : As the trains slow down, the amount by which each recoil increases the velocity of the fly is reduced. At the same time however, the amount by which the velocity of each train is reduced increases. Look at equation (4):

$$(4) v_{k+1} = (\frac{M-\varepsilon}{M+\varepsilon}) v_k - \frac{2\varepsilon(M-\varepsilon)}{(M+\varepsilon)^2} u_k + \frac{4M\varepsilon}{(M+\varepsilon)^2} w_k$$

Over time u_k increases enormously, v_k and w_k go down. Thus the middle term predominates. Until the trains begin to reverse their direction, u_k cannot decrease. This is confirmed by equation (3)

$$(3)\ u_{k+1} = (\frac{2M}{M+\varepsilon}) v_k - \frac{2M(M-\varepsilon)}{(M+\varepsilon)^2} w_k + (\frac{M-\varepsilon}{M+\varepsilon})^2 u_k$$

Observe that all three terms on the right hand side are positive **!**

The velocity of the fly peaks at its maximum when the polynomial $Q(u_n) \sim 0$. This has been shown to be in the neighborhood of

$$\boxed{u_n \approx \frac{50}{\sqrt{q}}!}$$

, which for this particular problem is about 30 million mph, an enormous figure but well below the speed of light. On the other hand, since in the final stage the speeds of the trains become extremely minute, one can imagine the fly batting away at a speed very close to this maximum for thousands of cycles.

Phase III: When the fly begins to reverse the direction of the trains, the velocities vk and wk are still governed only by the velocity of the fly, with the + and - signs reversed. In other words, if the velocities of the left and right trains respectively at some forward cycle are V and -W, while that of the fly is U, then in the reverse cycle the velocities of these trains will be -W and V, while that of the fly is still U. This follows directly from the conservation of momentum and energy.

Phase IV: As long as the trains are moving away from each other at velocities less than 50 mph, the flies velocity will remain above 90. However, something very strange happens when the train velocities are greater than or equal to 50. By the conservation of energy, (and also because the velocity of either the fly, left train or right train is sufficient to determine all the others), the fly's velocity will fall to 90 or below.

When this happens, the fly will strike either the right or the left train ***one more time*** **, then recoil with a velocity less than or equal to 10 mph.**

In the final phase, the 2 trains continue on forever moving away from each other at about 50 mph, while the fly trails along either to the right or the left, at about 10 mph.

Thus the system is neither periodic, nor does it die away in a whimper.

Finally we will show that , at any stage of this system, the difference $|v_k+w_k|$ is narrowly confined.

By the conservation of momentum, one always has:

$$M(v+w)+\varepsilon u = 90\varepsilon;$$
$$v+w=\frac{\varepsilon(90-u)}{M}=q(90-M)$$

The previous estimate on the maximum value of u is $u \approx \frac{50}{\sqrt{q}}$

Hence

$$|v+w| \le 90q + 50q\sqrt{1/q};$$
$$q \approx \frac{1}{4\times 10^{11}};$$
$$90q+50\sqrt{q} \approx \frac{90}{4\times 10^{11}} + (50\sqrt{10})10^{-5}$$
$$\approx (150)10^{-5}$$

Therefore the difference between the speeds of the two trains under the relentless pounding of the fly is of the order of ($1.5\text{x}10^{-3}$ mph.)