The Restricted Collision Theorem

Roy Lisker

November 22,2004

The Restricted Collision Theorem

Let N be a finite positive integer. Given N particles $p_1, p_2, \ldots p_N$, masses $m_1, m_2, \ldots m_N$, initial positions all on the same straight line $a_1 < a_2 < \ldots < a_N$, and initial velocities $\boxed{v_{x_1}, v_{x_2}, \ldots\ldots, v_{x_N}}$ *at time t = 0 , then the total number of all collisions between particles . both before and after time zero , is finite.*

Choosing coordinates

One can think of the N masses as beads on a wire extending infinitely in both directions. Our choice of reference frame is guided by the intention of setting the total momentum of the system to 0.

Having done that, we set the origin at the Center of Gravity (C.G. = "0") , so that the total Moment will be 0.

Having done this we write down the 3 conservation laws:

(1) Conservation of Moment:

$$\boxed{m_1x_1 + m_2x_2 + \ldots m_Nx_N = \sum_{i=1}^{N} m_ix_i = 0}$$

(2) Conservation of Momentum:

$$\boxed{m_1v_1 + m_2v_2 + \ldots m_Nv_N = \sum_{i=1}^{N} m_iv_i = 0}$$

(3) Conservation of Energy:

$$\boxed{m_1(v_1)^2 + m_2(v_2)^2 + \dots m_N(v_N)^2 = \sum_{i=1}^{N} m_i(v_i)^2 = 2E = const.}$$

A moment's reflection will show that the Restricted Collision Theorem is equivalent to the following pair of propositions:

P_1 : If one starts the system at time t = 0 and waits long enough, the configuration will assume the form:

$$\boxed{\leftarrow p_1 \leftarrow p_2 \dots . \leftarrow p_k ..(C.G.).. p_{k+1} \rightarrow \dots p_N \rightarrow \dots}$$

with particles moving away from the Center of Gravity in both directions without colliding ,

(so that $-v_1 \geq -v_2 \dots \geq - v_k \geq 0 \leq v_{k+1} \leq v_{k+2} \dots \leq v_N$)

P_2: In the infinitely distant past, either

(i) The particles were all stationary , or

(ii) The particles were infinitely far away, or

(iii) There was a Big Bang at some point! , or

(iv) The particles erupted spontaneously from empty space and began flying about (Steady State cosmology) , or

(v) We are living on the 3-D surface of a 4-D sphere. Particles fly into our vicinity, collide for awhile, then fly back to the opposite pole where they collide again, setting up an oscillating cycle that repeats forever. In this form of the Big Bang/ Big Crunch, the quasars at the far end of the universe should all be colliding with one another right now in a Big Crunch out there 14 billion light years away, sending them flying back to us to re-collide in a Big Crunch down here . Therefore the Big Bang and the Big Crunch are two sides of the same phenomenon, happening at opposing poles of the universe.

One cannot deny that there are some problems with (ii). Once particles come within finite distances of one another, the collision process will begin and end in a finite time interval after that. But there was no "time" at which particles were suddenly "only a finite distance away". Where were they before that time? If the relationships between particles are such that they were destined from the infinite past to collide in the present, that "destiny" must have been expressed in some finite distance and velocity relationships. Otherwise we are left with a "time before time" and a "space before space", which doesn't sit very comfortably.

❄❄❄❄❄❄❄❄❄❄❄❄❄❄

We now return to the proof of the Restricted Collision Theorem:

It is obvious for one particle, p_1 .

Two particles, p_1 and p_2 are either stationary, or move in opposite directions, or collide once and then move off in opposite directions . Since the origin of the rest frame is the center of gravity, the particles cannot move off together in the same direction.

The situation for 3 particles , p_1 , p_2 , p_3 is treated in the "Trains and Fly" paper.

Number the particles from left to right as p_1 , p_2 , p_3, p_4 ,...p_N Consider the interaction of p_1 and p_2 . The equations for their their velocities, before and after collision are:

$$\boxed{\begin{aligned} m_1 v_1 + m_2 v_2 &= m_1 v_1' + m_2 v_2' \\ m_1 (v_1)^2 + m_2 (v_2)^2 &= m_1 (v_1')^2 + m_2 (v_2')^2 \end{aligned}}$$

Since action equals reaction, an equal amount of momentum is exchanged between the particles , given by

$$\boxed{\mu = \pm\frac{2m_1m_2(v_1 - v_2)}{m_1 + m_2}}$$

We allow p_1 and p_2 to be moving in the same or opposing directions. The question we ask is: How much momentum is lost when the collision is so strong that p_2 will reverse direction, (or be immobilized) ? Three cases are distinguished:

(1) Both p_1 and p_2 are moving to the left. Therefore the velocities are both negative, and the velocity of p_2 must exceed that of p_1 in absolute value.

(2) Both are moving to the right. The velocities are positive and that of p_1 must exceed that of p_2 .

(3) They are moving in opposing directions, For there to be a collision, p_1 is moving to the right and p_2 to the left.

Case 1 : If the collision reverses the direction of p_2 , the mass of p_2 cannot be greater or equal to that of p_1 . Thus $m_2 < m_1$. Them for the reversal to occur, the momentum loss must exceed to momentum of p_2 , = m_2v_2 :

$$\boxed{\begin{array}{l} -m_2v_2 \le \dfrac{2m_1m_2(v_1 - v_2)}{m_1 + m_2} \\ -m_1m_2v_2 - (m_2)^2v_2 \le 2m_1m_2(v_1 - v_2) \\ m_1m_2v_2 - (m_2)^2v_2 \le 2m_1m_2(v_1) \\ (m_1 - m_2)v_2 \le 2m_1v_1 \end{array}}$$

Since it's been shown that m_1 must be larger than m_2 , we can divide through to obtain:

$$\boxed{v_2 \le \frac{2m_1}{m_1 - m_2} v_1}.$$

To calculate how must momentum loss this is in terms of v_1 , substitute back into the expression for μ :

$$\boxed{\begin{aligned} &\mu = \frac{2m_1m_2(v_1 - v_2)}{m_1 + m_2} \\ &v_2 \le \frac{2m_1}{m_1 - m_2} v_1 \\ &\therefore |\mu| \ge \left| \frac{2m_1m_2(v_1 - \frac{2m_1}{m_1 - m_2} v_1)}{m_1 - m_2} \right| \\ &= |v_1|\left(\frac{2m_1m_2(m_1 + m_2)}{(m_1 - m_2)^2}\right) = |v_1| K(1,2) \end{aligned}}$$

where K(1,2) is a function only of the masses of the two particles p_1 and p_2 . The importance of this equation lies in this: the amount of momentum loss of particle p_2 with each collision with particle p_1 is a function only of the masses of the two particles and the velocity v_2 . As each collision must increase the value of v_1 , it follows that the incremental loss of momentum of p_2 must also increase. Since there are only a finite number of particles, there is only a finite amount of energy in the system. Thus we have

Theorem 1: If p_1 is moving to the left, then total number of collisions of p_1 with p_2 must in which the direction of motion of p_2 is reversed is finite.

Corollary 1: The special case of the collision problem is provable from Theorem I in the situation in which p_1 moves to the left.

Corollary 2 : *(By symmetry)* : Likewise when p_N moves to the right.

Proof: **If p_1 ever begins moving to the left, the nature of the system is such that p_1 will never again move to the right. Each collision with p_2 will increase its momentum, and therefore cause a net loss of momentum to the system of particles to the right of p_1.**

Ultimately p_2 will cease reversing its direction when it collides with p_1. All other collisions are such that p_2 persists in the same direction as p_1. This is only possible if p_2 is being hit by p_3 . The total number of hits by p_3 which reverse the direction of p_3 must again be finite. Therefore it will continue in the same direction as p_1 and p_2 , and only hit p_2 when it is hit by p_4 . Continue inductively for particles p_5 , p_6 , ... until one comes to a particle p_k which doesn't go to the right and is not hit by anything to the right.

There must be such a p_k , since the total number of particles is finite, and p_N must always remain to the right of the origin.

Lemma: If p_1 moves to the left, there must come a time after which p_N also begins moving to the right.

Proof: **Obvious consequence of the Conservation of Moment. If p_1 moves out to infinity, when p_N remains at a bounded distance from the origin, the Moment must increase beyond limit. But the Moment is equal to zero at all times .**

By virtue of this lemma, one sees that a time will come when all the particles to the left of the origin are moving to the left, and all the particles to the right are moving to the right. They may collide a few times in the same direction. It is easy to see that these collisions will die away.

One can also reason as follows: the streaming away in both directions of the system from the origin divides it into two systems, each of which have fewer particles than the original. By induction on the number of particles, and the fact that its been proven for N = 1,2 and 3, the theorem is true for the sub-systems and thus for the combined system .

The situation which develops when p_1 is moving to the right is analogous. However, there are certain peculiarities in this case which are not present when p_1 is moving to the left.

If p_1 is moving to the right, then

(i) It eventually reverses direction, which means than p_N also eventually begins moving to the right, and we have the conditions of Theorems 1 and 2.

(ii) It does not reverse direction, but receives an infinite number of hits from p_2 before coming to a halt.

This case can be dismissed through reasoning similar to that used in proving Theorem I. For p_2 to hit p_1 several times without p_1 reversing direction, p2 must reverse direction between each hit. If both p_1 and p_3 are moving towards p_2 , then the increments in the momentum of p_2 will continually increase. There are only two ways that this can stop. Either p_2 will reverse p_1 , creating the situation of Theorem I, or p_2 will reverse the

direction of p_3 , so that p_1 , p_2 and p_3 are all moving in the same direction.

Proceeding inductively, one sees that eventually p_1 , p_2 ,... p_k are all moving towards the origin from the left, and p_{k+1} ,...p_N towards the origin from the right.

When the two sets S_L and S_R of left- and right- particles collide, they will hit with equal and opposite momentum, flying off simultaneously at velocities equal and opposite to those they had at the moment of collision.

Summarizing:

(1) If p_1 moves to the left or p_N to the right of the origin, then Theorem I and its corollaries show that the system will eventually turn into a stream of particles moving to the left and another stream to the right; in fact

(2) If p_1 moves to the left, then pN must eventually move to the right of the origin

(3) If p_1 moves towards the origin then, either:

(a) It eventually reverses direction under the effect of collisions with , and therefore p_N eventually moves to the right.

(b) It moves all the way to the origin. In that case, all the particles arrive at the origin simultaneously, there is a great explosion and they stream away in the configuration described in (1).

There are of course other possibilities. It is possible for a collection of particles surrounding the Center of Gravity to remain

stationary or become stationary. The remaining particles will still stream off in both directions.

Of course it is not ruled out that all particles are stationary and have been so from the beginning of time.

Simultaneous collisions of 3 or more particles can occur without having to modify these results.

❇❇❇❇❇❇❇❇❇❇❇❇❇❇
❇❇❇❇❇❇❇❇❇❇❇❇❇❇