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Time Derivatives and Uncertainty

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SUMMARY

It is shown that the conceptual structure of the Quantum Theory is such that the time derivatives of all quantities have an intrinsic uncertainty; in some sense they are non-commuting with themselves. This sets limits to the application of Newton's Laws of Motion to the microscopic realm. One way of interpreting this is to introduce a "time-dependent" momentum,(in analogy to "time dependent energy"), in addition to the "time-independent" momentum in customary Quantum Theory.

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Introduction

Only a restricted set of classical physical magnitudes are amenable to being "quantized" by the standard operator schemas of quantum theory. We did not need Einstein to tell us that God doesn't play dice to reveal the incompleteness of Quantum Theory to us. [1]

This schema for quantization, which we may acronymize as the "O.S." , is employed as a mechanical method for "translating" certain classical, Newton - Lagrange-Hamiltonian equations of physics into a quantum operator which can be applied

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[1]God of course doesn't need to play dice: HE runs Las Vegas.

to the Schrödinger wave function, ψ, to produce a probabilistic or quantum description.

The physicist readers of this article will be familiar with this schema:

The Operator Schema (Quantization)

$$Position: x, y, z \rightarrow x\bullet, y\bullet, z\bullet : (\psi)$$

$$Momentum: p_x, p_y, p_z \rightarrow (ih/2\pi)\partial()/\partial x, (ih/2\pi)\partial()/\partial y,$$

$$(ih/2\pi)\partial()/\partial z : (\psi)$$

$$Energy: E \rightarrow (-ih/2\pi)\partial()/\partial t : (\psi)$$

Those persons not familiar with the Operator Schema , (henceforth abbreviated as the 'O.S.'), may not be able to understand much of the rest of this paper. However, it is not required that one knows *much more* about Quantum Theory than this , (and the Uncertainty Principle), to be able to read it for pleasure, profit, insight, shock , and so forth.

The *Inverse Schema* , (I.S.), may also be given a meaningful interpretation. To "*continuize* " [1] a quantum equation is equivalent to the passage, via the Correspondence Principle , from the sub-atomic level to the macroscopic everyday world. (For persons not familiar with the way quantum mechanics works I might explain that, theoretically at least, one can take any differential equation that purports to describe the physical world, (mechanical, gravitational, hydrodynamic, electrical, etc.), replace every magnitude in the equation by its equivalent operator in the O.S., and thereby derive an equation which, in some peculiar way that is still a matter of hot debate, describes the microscopic realm.)

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[1]Coinage of the author's.

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The Inverse Schema
(Correspondance Principle)

$$Position: x\bullet, y\bullet, z\bullet : (\psi) \rightarrow x, y, z;$$

$$Momentum: (ih/2\pi)\partial()/\partial x, (ih/2\pi)\partial()/\partial y,$$

$$(ih/2\pi)\partial()/\partial z : (\psi) \rightarrow p_{x,} p_{y,} p_z;$$

$$Energy: (-ih/2\pi)\partial()/\partial t : (\psi) \rightarrow E$$

The O.S. and the I.S. are most often applied in the translation of the Hamiltonian, or energy equation, back and forth from Classical to Quantum mechanics .In its simplest form, the Hamiltonian is the first integral of Newton's 3rd Law of Motion:

$$(1)\ (1/2m\ p^2) + V(x, t) = E$$

We pass this expression through the O.S. to produce the famous time-dependent Schrödinger equation:

$$(2) -((h/2\pi)^2/2m)\nabla^2\psi + V\psi = -(ih/2\pi)\partial\psi/\partial t$$

A generalization of the same method produces the Klein- Gordan equation, which, in its turn, ingeniously factored, produces the Dirac equation for a free electron. If one then attempts to apply the I.S. to the Dirac equation, the results are not very satisfactory; yet, by the use of a bit of "inverse ingenious trickery" , one can get out the classic relativistic relationship of energy to momentum:

$$(3)\quad E^2 = c^2 (p^2 + m^2c^2) .$$

When there is no reason for using the actual values of the speed of light and Planck's constant, we will choose our units in such a way that $c = h/2\pi = 1$.

Despite their proven power in terms of creating a vision of the sub-atomic domain , as an aid to our thinking , and as a way of making calculations and predictions, the O.S. and the I.S. fail

dramatically when applied to those equations which are most fundamental to all of physics, ***Newton's Laws of Motion:***

(i) $p = m\, dx/dt = mv$

(ii) $F = m\,\alpha = dp/dt$

The failure of the O.S. when applied to equation (ii) in particular means that the Newtonian vision of the cosmic order breaks down completely when applied to the quantum domain. This observation, which may be obvious to some, yet which is far from elementary, means among other things that any viable form of Quantum Gravity must await a substantial improvement in our mathematical description of the structure of the cosmos.

Let us examine equation (ii) first: The O.S. is applied in such a fashion that a string of multiplications on the "quantitative" side is translated into a chain of functional iterations on the "operator side:

(a) x^2 ----------> $X^2 \bullet / \psi$

(b) $p_x^2 = m^2\,(dx/dt)^2$ ----> $i\partial\ /\partial x\ (i\,\partial(\ \ )/\partial x) :(\psi)$

$= -\,\partial^2(\ \ )/\partial x^2\ :(\psi)$

(c) $x \cdot p$ -------> $ix\,\partial(\ \ )/\partial x :(\psi)$

(d) px-------> $i\,\partial(x\ )/\partial x = ix\partial(\ \ )/\partial x + i \bullet :(\psi)$

The last two relations, (c) and (d) show that the O.S. is not even, strictly speaking, even linear , since subtracting (c) from (d) on both sides gives us :

(4) 0 ---------------------> i

:A rather striking way of stating the Uncertainty Principle!

Clearly ***any*** **application of the O.S. to acceleration must be ambiguous. What happens when we attempt to quantize**

$\alpha = d^2x/dt^2$ **? No iterations of the differential form , d/dt on the left side have any interpretation in terms of either iterations or multiplications of partial differential forms on the** right. Certainly, the continuizing of $\partial^2(\ )/\partial t^2$ by the I.S. will not yield anything remotely resembling acceleration:

$$\partial^2(\ )/\partial t^2 \xrightarrow{\text{(O.S.)}} -E^2 \quad (!??)$$

As for equation (i), it presents special problems since *momentum* , like *space* and *time* , is an irreducible primitive in the quantum theory.[1] *Mass* is a parameter, while dx/dt is the ratio of two commuting quantities, x and t, and therefore, in theory, not restricted by the Uncertainty Principle.

From the standpoint of quantum theory a better form of equation (i) i s :

$$\text{(iii)} \quad m = p/(dx/dt)$$

In this equation the right side contains the 3 primitive notions, *space, time* and *momentum* , while *mass* is defined indirectly in terms of these.

However: we are going to take a second look at this situation . Through a return to the conceptual images that lie at the basis of all quantum descriptions , we will discover that *there is a measurable uncertainty associated with the velocity v = dx/dt, dependent only on the uncertainty in t* . We will then apply this perspective to acceleration. The results are quite interesting.

II. Taking The Time Derivative

The *Ansatz* for finding the first derivative of a function ,x(t), is familiar to all of us since the time of Isaac Newton. Letting t be the independent variable, one chooses two moments very

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[1] In contrast, for example, to the situation of Special Relativity , where momentum is a derived quantity.

close to one another, say t and t+ ε . After measuring the two quantities x(t) and x(t+ ε) , one forms the ratio

$$(5) \qquad [x(t+\varepsilon) - x(t)]/\varepsilon = \Delta x / \Delta t$$

A pure mathematician can imagine that the time increment Δx = ε ---> 0 ,to derive the exact form of the derivative. In the (classical) physical world however, it is sufficient to choose ε to be so small that any further reduction may be treated as negligible. The success of this approximation depends in turn on assumptions about the continuity, differentiability and degree of smoothness of the function x(t) . Thus pure mathematics gives way to approximate mathematics, which in its turn devolves on matters of pure mathematics. As the woman said in the famous retort to Bertrand Russell: " It's turtles all the way down!" [1]

Yet the conceptual images of the Quantum Theory are radically at odds with this methodology . *The very act of taking the initial measurement , at time t, of the location x of some particle Q, of mass m, so perturbs both the position and the momentum of Q, that the second measurement at x (t+ ε) , cannot possibly have any continuous relationship to x (t) !*

The situation is not hopeless, because the Uncertainty Principle gives us a way of estimating the size of this perturbation. Let's say that the initial measure of x(t) has created displacements in the position and momentum of Q by the amounts dx and dp .

By the Uncertainty Principle we have:

(6) dxdp ≥ 1/2 (*Assuming, as usual, h/2π =1* **)**

After a time interval , ε, the uncertainty in the displacement , x (t+ ε) , will contain a term for dx , and also a term D derived

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[1]**See Stephen Hawking's thoughtful and funny book, "Brief History of Time"; Bantam,1988**

from the uncertainty in the momentum which is given by $D = (\varepsilon\, dp)/m$.

We therefore write:

(7) $x^{(act)}(t+\varepsilon) = x^{(hyp)}(t+\varepsilon) + dx + D$,

where $x^{(act)}$ is the estimated actual value of the new location, while $x^{(hyp)}$ is the value it *would have had* if the first measurement had not been taken. (For the present discussion we ignore the value of $d\,x^{(hyp)}(t+\varepsilon)$.

One has:

(8) $x^{(act)}(t+\varepsilon) = x^{(hyp)}(t+\varepsilon) + dx + D$ = $x^{(hyp)} + \{dx + \varepsilon\, dp/m\} \geq x^{(hyp)} + dx + \varepsilon/(2m dx)$

, the final term on the extreme right coming from the Uncertainty Principle .Form the differential expression for x :

(9) $\Delta x/\Delta t = (1/\varepsilon)(x^{(act)}(t+\varepsilon) - x(t))$

$\geq 1/\varepsilon(\ x^{(hyp)}(\varepsilon) - x(t)) + D/\varepsilon$

which is a first approximation to $dx/dt + \{ dx/\varepsilon + 1/2m dx)$

$= dx/dt + \mu$,

where $\mu = dx/\varepsilon + 1/(2m dx)$ will be called the *intrinsic uncertainty in the velocity ,(or time derivative*).

Let $dx = z$, and calculate the minimum for $z > 0$, of the expression

$\rho = z/\varepsilon + 1/(2mz)$.

Taking the derivative one obtains:

(10) $\rho' = 1/\varepsilon - 1/2mz^2 = 0$

Solving this gives $z_{min} = \sqrt{(\varepsilon/2m)}$, and, substituting in ρ,we get ,finally:

(11) $\rho_{min} = \sqrt{(2/(m\varepsilon))}$.

For the statement of our first theorem reintroduce $h/2\pi$:

THEOREM I: The uncertainty in the expression,

$$(x^{(act)} - x(t))/ \varepsilon = \Delta x/\Delta t ,$$

is at least $\mu = \surd (h/\pi m\varepsilon)$.

As ε --------> 0 , this expression goes to ∞ . The precise time derivative at the initial moment, t, must therefore be acknowledged to be unknowable. If ε is too small, μ will become too large, and if ε is too large, then x will be too far away from the first derivative to be of any use. We can actually estimate, for a given situation, the best value for ε :

THEOREM II: The best estimate for the increment ε is found at the place where the product of the differences of x from the hypothetical values of x ′ at t and at t+ ε , is equal to the square of the uncertainty, μ .

PROOF: We seek to minimize the expression

$$(12) \qquad \Sigma : \quad (x'(t+\varepsilon) - \rho)^2 + (\surd(h/(\pi m\varepsilon))^2$$

, over the variable ε. .

Here, $\rho = (dx/dt)^{(hyp)}$ is the time derivative which we assume to "exist" in some sense, in a pre-measurement reality.

Then,

$$(13) \quad 1/2 d\Sigma/d\varepsilon = (x'-\rho) x'' + \mu \mu' = 0$$

$$\mu\mu ' = -\hbar/ m \varepsilon^2$$

Assuming that ε^2 will be conveniently close to zero, we invoke the 'ideal approximations' :

(a) $x''(t+\varepsilon) \approx -(1/\varepsilon^2) [\varepsilon x'(t+\varepsilon) - \{x (t+\varepsilon) - x(t) \}]$;

(b) $x'(t+\varepsilon) - \rho \approx [\{x(t+\varepsilon) - x(t) \}/\varepsilon] - x'(t)$

Juggling the terms around a bit leads to the expression:

$$(14) \qquad (x - x'(t+\varepsilon)) (x - x'(t)) = h/(2\pi m\varepsilon)$$

, which proves the theorem.

This result is somewhat academic , as it implies that one must know the theoretical behavior of x′ in this part of its domain in order to derive a "best value" for ε., i.e., that which gives the best approximation to x′ at t. The theorem does show however, that a "best value" does exist when the above equation can be solved for ε. Since so much of quantum theory is based on thought experiments, this situation frequently arises. For example, if one makes the assumption that the displacement, x , is a quadratic expression in the time , t, then equation (14) becomes linear in ε on the left side. Solving the resulting equation gives at most 2 values for ε . These values might be used to test our hypothesis of a quadratic dependence of x on t . Note that this is the best test that we can make of this hypothesis.

III. Acceleration

Epistemologically, the correct form for Newton's 3rd Law of Motion is

(15) $F = dp/dt$

Only rarely does *mass* , per se, enter nakedly into physical theory. The concepts of *energy*, *momentum* and inertia are far more fundamental .Any measurement of mass is dependent on them . They are also the quantities which our senses record as intensities , which, on the basis of various theoretical assumptions ,are then interpreted to give a parameter for mass.

This is very convenient for the treatment given in this paper. Attempting to deal with the uncertainty relationships in an expression such as

(16) $dv/dt = [\, d\,(dx/dt)/dt\,]$,

is bound to run into serious complications, whereas the *intrinsic uncertainty* in the time derivative of the momentum

dp/dt can be treated directly in a manner homologous with the previous estimation of the uncertainty in the velocity.

Proceeding as before , an initial measurement of p at time t will have a margin of error dp , and an additional displacement error of dx . After performing a second measurement at time t+ ε , we add on these two displacements ,

(i) dp , and

(ii) a term $\lambda = mdx/\varepsilon$. Then,

(17) $$G = (p^{(act)}((t+\varepsilon) - p(t))/\varepsilon \geq \{(p^{(hyp)}(t+\varepsilon) - p(t))/\varepsilon\} + \{(dp + m\,dx/\varepsilon)/\varepsilon\} = \beta + \phi$$

where β is the measurement of the acceleration, and φ is its uncertainty. Once again we apply the Uncertainty Principle to convert the final term into an expression in dp . Let ω = dp . We wish to minimize $\phi = dp/\varepsilon + m/2dp\,\varepsilon^2$ over ω . This leads to :

(18) $\phi' = 1/\varepsilon - m/(2\varepsilon^2\omega^2) = 0$ **. Solving:**

(19) $\omega_{min} = \sqrt{m/2\varepsilon}$**, and** $\phi_{min} = \sqrt{(m/2\varepsilon^3)}$**.**

We once again reintroduce the letter, h/2π, and express our result in terms of the force, F = mα

THEOREM III : The uncertainty in the determination of force at the quantum level, over a time interval ε , is given by $\phi = \sqrt{(mh/4\pi\varepsilon^3)}$. The uncertainty in the acceleration is therefore(1/m) $\phi = \sqrt{(h/4\pi m\varepsilon^3)}$.

Once again, we seek "best values " for β and ε . Our method is homologous in all respects to that used in deriving the equivalent value for the velocity. The result is:

THEOREM IV : The product of the differences of β from the hypothetical values of dp/dt at t and at t + ε , is equal to 3/2 x φ, the Uncertainty .

#11.

Interpretation

One may interpret these results by positing two kinds of quantum-theoretic momentum*: time-independent and time-dependent* . This has the effect of:

(i) Putting the momentum on a par with the energy, which also has time-independent and time-dependent forms[1], and

(ii) highlighting even further the incurable ambiguity of the status of time in the quantum theory. Depending on how it is used, time is either (i) a parameter, or (ii) a variable commuting with space but not with energy.

(A) ***Time-independent momentum:*** **This is the form which can be in some sense measured instantaneously : an electron smashing into a plate and leaving some kind of indentation, or a blackened spot on a photographic film, the intensity of the blackening being a measure of the instantaneous momentum of the collision.**

(B) Time-dependent momentum : **This is the quantity that occurs in the expression $p = mv$,(or $p = mv/\sqrt{(1-v^2/c^2)}$ .It is not measured by some kind of instantaneous impression on a recording medium, but through taking a reading of the positions of a particle of known mass at two infinitesimally close points in time.** ***It is this velocity which has an intrinsic uncertainty*** **.**

These distinctions should find some application in Special Relativity as well: here time-independent momentum is something of a fiction, as it may be modified, even removed entirely, by changing the reference frame of the impinging surface of contact.

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[1]Recall that energy and momentum are linked through Special Relativity as the temporal and spatial components of the same 4-vector.

Time-dependent momentum, however, is reference-frame invariant, like zero-point energy, which it resembles to some extent.

References

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