Galilean Relativity and the group structure of Inertia

Roy Lisker
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rlisker@yahoo.com
www.fermentmagazine.org

[1] ***The requirements for an inertial group, that is to say, a group of inertial motions on a homogeneous manifold:***

For the moment we assume that O sees himself as stationed on a 2-dimensional surface, M_O, where the subscript signifies that someone else moving along this surface might see something different ; O himself can be "3-dimensional' as we are, but all of his measurements are on M, even as we measure distances and motions on the earth.

(a) In Galilean relativity there is no "absolute rest". There are enough problems with this that one does not have to go to special or general relativity for a long time to come.

Since there is to be no "absolute reference frame", there must be criteria whereby an observer O can determine that he is on an inertial path. The classical definition by Einstein is the following:

O is assumed to be *at rest* in an inertial reference frame , *if he is unable to detect his own motion* ,

Pragmatically, this means that he is equipped with a collection of motion detectors which collect data of some sort indicating the absence or presence of motion. We select 3 out of a host of possible indicators of motion:

(1) The local curvature of space under goes no change. O draws a triangle around his feet made up of sides of a standard length, say a 3-4-5 triangle T which would be a right triangle in flat Euclidean space, but have other angles on a sphere. Periodically he measures the sum of the angles of T. If this changes then he concludes that he is not in an inertial frame

(2) There is no force either pushing him or acting on him. This can be detected either as

(i) Heat

(ii) A gravitational attraction.

In the absence of these O will assume that he is moving in an inertial frame.

It is clearly easier to see what is going on, if one assumes that there is an n-dimensional manifold M "embedded" in a higher dimensional manifold M* in which M is perceived to be unmoving. If O is a particle on M, and I is another witness ("myself") in M* , then we can define an inertial motion as follows:

(1) M is homogeneous. There is a congruence carrying any point on M into any other point on M. It "looks the same" to I when seen from any vantage. For example, if M* is 3-space, then the homogenous manifolds would be the plane, cylinder, sphere, straight line, helix and circle.

From "I's vantage", any observer O moving along a geodesic path on one of these at a uniform velocity v would imagine himself at rest. But what does I know of what O thinks? The key is in the time dimension. If U and V are moving along the length of a helix, and, to I, U moves at velocity u, V at velocity v, both can adjust

their measurement of time by altering the speed of the hands of their clocks, so that each covers "one unit of space" in "one unit of time". By relating only to I they might imagine themselves both to be at rest. It is only when comparing their paths with one another that they detect "relative motion".

(b) Assuming that the determination that one is "at rest in one's own reference frame" has been made, we ask for the conditions for an inertial motion , and for an ***inertial group*** **of motions .**

Let A and B be two observers, each of which determines himself to be "at rest". In an "inertial dynamic space", if A and B see themselves as being at rest, then each will see the other as moving ***along a geodesic*** **at a** ***uniform velocity*** **. If A sees himself at rest, and observes that either**

(i) B is not moving at a uniform velocity; or

(ii) B is not moving along a geodesic ;

(iii) or both

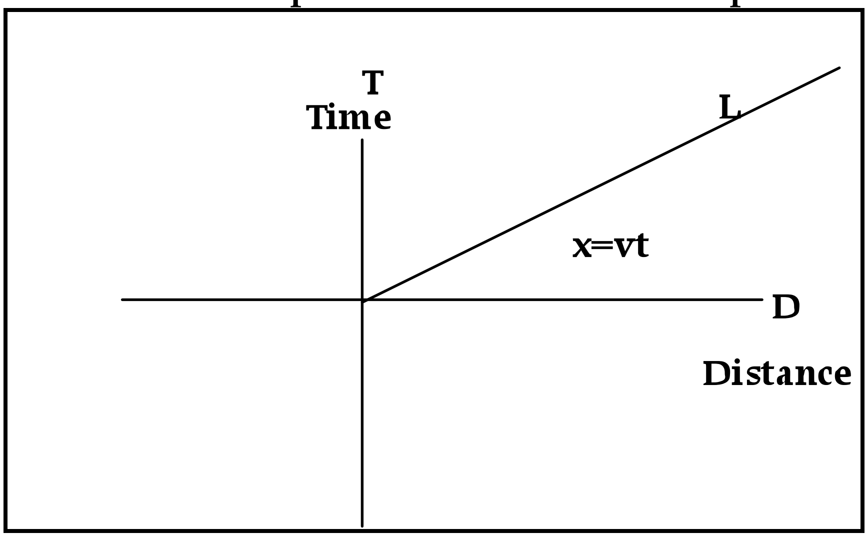
Then B must detect evidence that he is accelerating **.**

(c) The inertial group. This demands the following conditions

(i) Suppose A observes that B is moving along a geodesic at a uniform velocity v. Suppose there is another observer C, when B observes to be at rest relative to himself. ***Then A will observe C to be moving at v*** **, (where v is a n-vector velocity) . This condition allows one to speak of "reference frames".**

(ii) Suppose A observes B moving along a geodesic at velocity u and B observes C moving along a geodesic at velocity v. ***Then A will observe that C moves along a geodesic at velocity w = u (+) v, where (+) is a group operation with identity 0, and inverse -u for u, etc*** **.**

In Euclidean geometry, time is a dimension outside the metric. However, in Galilean Relativity it is metrizable with respect to the spatial metric. This makes possible the notion of a uniform velocity, v. Consider a "1 dimensional" Galilean space, with metrizable time coordinate. The Cartesian representation is an affine space.



Through altering the ratio of the space unit 1_t to the time unit 1_s in the ratio of proportionality k, a "velocity" v = x/t , is shifted to another velocity
v' = kx/t. This is neither a rotation nor an isometric transformation but what can be called a "similarity transformation": a change in velocity can be interpreted as a change in units, and the reference frame on L "sees" the world exactly the way the world is seen by the frame on D.

Needless to say, this can easily be cast in the form of "topological invariance", so that for example, motion along any infinite, non intersecting arc in space can be mapped diffeomorphically in such a way that the inertial group on R is preserved.

(2) The inertial group in the pseudo-Euclidean or Minkowski space is unusual in that it is not a similarity group, but an isometry connecting

all 4 dimensions of x,y,z, t. Linear trajectories become linear trajectories under a Lorentz transformation, but the group law is very different:

$$\boxed{v_1(+)v_2 = v_3 = \frac{v_1 + v_2}{1 - {v_1 v_2}/{c^2}}}$$

There appears to be nothing to prevent the combination of both forms of inertial group in a group product which applies over a 3-dimensional space with coordinates (x, r, t): The group for x, t is the Galilean group on Euclidean space; the group for x, r is the Lorentz group on pseudo-Euclidean space. The metric on the "x,r" plane is $\boxed{-ds^2 = dr^2 - dx^2}$. There is no metric on the x,t plane, but the similarity group creates equivalent Galilean reference frames. If L is the Lorentz group, G the Galilean group, then the group governing this universe is

$$\boxed{H = G \otimes L}$$

Explicitly, one must posit a "compound velocity" v = (u,w). Observers A,B, C. A observes that B moves at velocity $v_1 = (u_1, w_1)$; B observes that C moves at velocity (u_2, w_2). Then A observes that C moves at compound velocity v_3, where $\boxed{v_3 = (u_3, w_3) = (u_1 + u_2, \frac{w_1 + w_2}{1 - {w_1 w_2}/{c^2}})}$

This construction implies the possibility of two time dimensions. In a later part of this paper we will show that there is no contradiction between this and traditional Newtonian mechanics. Indeed, there may be a model for this phenomenon in the "cosmic time" of the expanding universe. "Red shifts" of frequency are assumed to combine linearly, while mechanical motions are governed by the Lorentz Group.

One runs of course into problems with the notion of distance. If a system is observed to move away from the observer O, with a compound

velocity v = (u,w), how "far" has he "gone" after one hour? Answer: how do cosmologists and astronomers solve this? Perhaps simply D = (u+w)t. Perhaps distance is an illusion produced by time, energy and the expanding universe.

One can even propose a model for two different sorts of "clocks" to "measure" the two kinds of time. Velocities in the "cosmic time" are measured by red shifts,. The regular pulsation of light frequencies function effectively as quantum clocks.
The quantities of "mechanical time" are measured by the standard distances traversed by a light beam. Thus "light trajectory" and "light pulsation" are the two clocks operating in the two time dimensions.

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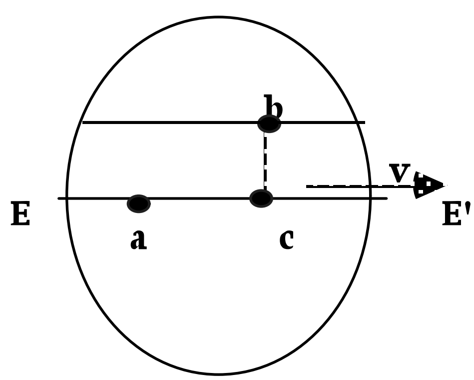
One is naturally interested in exploring all the possibilities inertial groups, and for manifolds that support them. An important negative result is that one can't place an inertial group structure on a spherical surface with standard metric:



S is a spherical surface with equator EE'

a is "at rest in its own frame" on the equator . c is observed to move away from a along EE' at uniform velocity v. S being the "ambient space", the universal curvature is the same everywhere. c experiences no forces of either friction or a Coriolis nature; the angle sums of standard triangles do not change.

Let b represent another location on the sphere between the equator and the North Pole, where there is an observer "at rest" in the reference frame of c.

Both b and c observe a to be moving towards the West at a uniform velocity v. *However a sees b moving at a different velocity, along a curve that is not a geodesic* **! Let the radius of the sphere be R. a observes that b makes an apparent circle around the globe with a radius r < R. Since the time for a complete circuit of c is the same as the time for a complete circuit of b, it follows that c covers a distance of length 2πR in the same time that b covers a circuit of length 2πr . It follows that the apparent velocity of b from the vantage of a is u = v(r/R). From these observations a concludes**

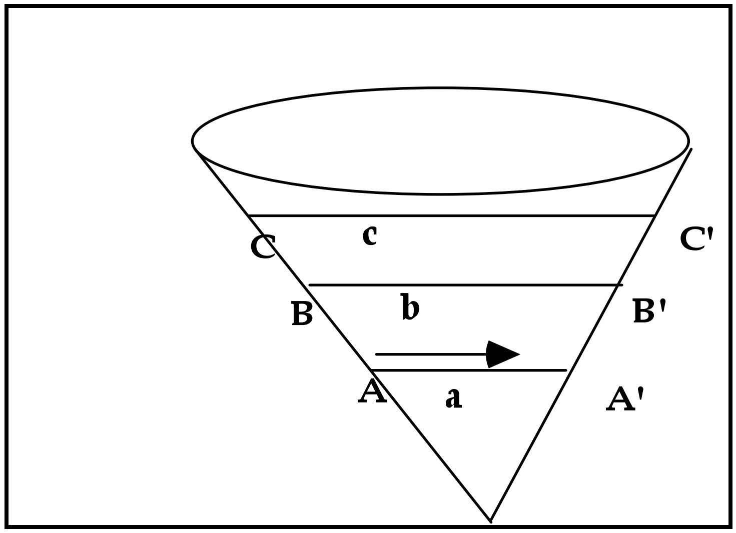
(1) b is not in the same reference frame as c. In fact

(2) b is not even moving along an inertial path

At the same time, b and c have no difficulty in concluding that they are in the same reference frame. One cannot argue that such circuits be included into the class of "inertial trajectories" unless one wishes to exclude great circles altogether. An inertial path is completely determined by the Lie algebra element, or initial direction, in which a particle begins moving: it cannot move in two directions at once.

One might be tempted however, to see if a structure of "inertial trajectories" could be constructed on a cone, on which *only one* **direction**

is permitted at each point; namely the motion along a circle perpendicular to the axis of the cone:



Similar arguments show that, when b and c see each other as being at rest,
a "moving observer" along AA' will observe them to be moving at different velocities.

One can of course place a Galilean structure on the cone in which the inertial paths are the conical geodesics. The conic surface is "flat", having zero curvature everywhere, and the local geometry is always Euclidean . The combination of Galilean relativity with self-intersecting geodesics may produce some exotic dynamical systems , worth pursuing in another paper.

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