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Iterate Sets and Patterned n-Cimals

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Introduction

An n-cimal is the representation of some real number to the base n. For n=10 this is the customary decimal representation. If α is a positive real number less than 1 , then $\alpha_{[n]}$ signifies the sequence of digits employed in its representation, base n . In general this will be notated as :

(1) $$\boxed{\alpha_{[n]} = {}_n\varepsilon_1^\alpha \, {}_n\varepsilon_2^\alpha \, {}_n\varepsilon_3^\alpha \cdots {}_n\varepsilon_k^\alpha \cdots}$$

When the base is clear from the context we shall, in general, drop the pre-subscript "n". For all of the theorems in these papers, the specific value of the base is not relevant.

The iterate collection , or iterate set $\boxed{I_n^\alpha}$ **of sequences associated with the n-cimal representation of α is obtained by shifts on the n-cimal point to the right, throwing away the integer part and keeping the remainder. To be precise :**

$$\boxed{\begin{array}{l} I_n^\alpha = \{\alpha_k\}, \mathit{where} \\ \alpha_k = \alpha n^k - [\alpha n^k] \end{array}}$$

A patterned n-cimal α **relative to a fixed base n is any real number which is a limit point of its own iterate collection. If one of the iterates of α is a patterned n-cimal, then we will say that α is *effectively patterned* relative to n. This paper will investigate the properties of the boundary, or closure, of the iterate collections of real numbers. In particular we will be looking at such questions as :**

(i) What are the conditions on α so that $\boxed{\overline{I_n^\alpha}}$ **is finite? Countable? Uncountable**

(ii) What is the Lesbesgue measure of $\boxed{\overline{I_n^\alpha}}$?

(iii) When is $\boxed{\overline{I_n^\alpha}}$ a perfect set?

(iv) Properties of the iterate collections of rational numbers, algebraic numbers and other interesting classes of numbers.

The property of being a patterned n-cimal is equivalent to this following: Let $\boxed{L_{j,k}^\alpha}$ stand for the segment of the n-cimal representation of α from the j^{th} to the k^{th} digit. Then, for each such block, there are infinitely many identical blocks $\boxed{L_{m_i,m_i+k-j}^\alpha}$ at starting points $\{m_i\}$ in the n-cimal representation. This follows immediately from the algorithm through which $\boxed{I_n^\alpha}$ is derived and the fact that α is a limit point of the iterate collection.

Rational numbers may be considered effectively patterned n-cimals, although the iterate collection is finite. n-cimal representations of rational numbers, to any base, become periodic after a certain point.

The simplest example of an irrational patterned n-cimal β is the following: Let n = 2. If h is any positive integer, write it as $h = k2^m$, where m is the exponent of 2 in h. Let B_h be a block of 1's of length m . If m = 0, then let $B_h = 0$. The expression:

$$\boxed{\begin{aligned}\gamma &= .B_1B_2B_3\ldots\ldots B_k\ldots\\ &= .0101101011101011010111 10\ldots\end{aligned}}$$

is a patterened n-cimal to the base 2.

Theorem I :

If there is even one block of the form $\boxed{B = \lambda_1\lambda_2\ldots.\lambda_k}$, where the entries are taken from the elements of Z_n = (0,1,2,,...n-1) and k is finite , which does not appear anywhere in the n-cimal expansion of a positive real number β , then the closure of the iterate set $\boxed{\overline{I_n^\beta}}$ of β is a set of measure 0.

Proof:

Let C(B) stand for the collection of all n-cimals which *do* contain the block B somewhere along the line. Since B appears nowhere in , it will not appear in any of the n-cimal expansions of the numbers of $\boxed{\overline{I_n^\beta}}$. It follows that $\boxed{C(B) \cap \overline{I_n^\beta} = \varnothing}$.

C(B) has positive measure: for example, the set of all n-cimals which have B as head block has measure $1/n^{k+1}$. In the same way the intersection of C(B) with every sub-interval of [0,1] has positive measure. By the Lesbesgue derivation theorem C(B) has measure 1. Since $\boxed{C(B) \cap \overline{I_n^\beta} = \varnothing}$ the theorem follows.

Corollary :

If $\boxed{\mu(\overline{I_n^\beta}) = 1}$, then every finite block B of ordered sequences of elements of Zn occurs in β infinitely many times.

Theorem II:

If $\boxed{\rho \varepsilon \overline{I_n^\beta} \ then \ \overline{I_n^\rho} \subseteq \overline{I_n^\beta}}$

Proof:

If ρ is in $\boxed{I_n^\beta}$, then the closure of its iterate set coincides with that of β . If ρ is in $\boxed{\overline{I_n^\beta}}$ but not in $\boxed{I_n^\beta}$, that is to say it is not an iterate of β , then any block of digits occurring in ρ must occur in β infinitely many times. If ψ in $\boxed{\overline{I_n^\rho}}$ is one of this sets limit points, then any block occurring in ψ must also occur in ρ infinitely many times, therefore also in β infinitely many times . It follows that ψ is also a limit point of $\boxed{I_n^\beta}$, hence by definition an element of $\boxed{\overline{I_n^\beta}}$. Q.E.D.

Theorem III (Fundamental Theorem) :

If β is any real number , then, for any base n , the set $\boxed{\overline{I_n^{\beta}}}$ contains at least one patterned n-cimal.

Proof:

Let $\boxed{P_n^{\beta}}$ be the set of all the limit points of $\boxed{\overline{I_n^{\beta}}}$. We proceed by indirect proof and assume that the theorem is false. Then in particular, b itself is not patterned , and we have $\boxed{I_n^{\beta} \cap P_n^{\beta} = \varnothing}$. Choose an element $\boxed{\rho_1 \in P_n^{\beta}.}$ By assumption ρ_1 will also not be patterned and one has, again $\boxed{I_n^{\rho_1} \cap P_n^{\rho_1} = \varnothing}$. Since $\boxed{I_n^{\rho_1} \subset P_n^{\beta}}$ (every iterate of a limit point must also be a limit point), one also has $\boxed{I_n^{\beta} \cap I_n^{\rho_1} = \varnothing}$. Proceeding in this fashion we obtain a sequence of numbers, $\boxed{\rho_1, \rho_2, \ldots, \rho_k, \ldots}$, with corresponding sets $\boxed{I_n^{\rho_j} \cap I_n^{\rho_k} = \varnothing}$, $\boxed{I_n^{\rho_j} \subset P_n^{\rho_{j-1}} \subset \overline{I_n^{\rho_{j-1}}}}$.

Since $\boxed{\overline{I_n^{\beta}}}$ is compact, the above sequence of numbers converges to a non-vacuous set of limit points, all in $\boxed{P_n^{\beta}}$. Every one of these limit points is patterned or eventually patterned. QED

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The following question is of interest: What are the general conditions on an arbitrary real number α , in base n representation, so that the closure of its iterate set will be finite, countable or uncountable? A few examples:

If α is any rational number, then its iterate set to any base n will be finite, therefore discrete. The converse is also true, as a finite iterate set can only be created by an n-cimal that is periodic after a certain point, which always converges to a rational number.

The following example to the base 2 has a countable iterate set:

λ = 0.101001000100001000001000000100000001.........

The points on the boundary are .1 , .01 , .001 , .0001 , ...

There is no connection between a countable iterate set and computability. Replace all the 1's in the above sequence with some non-computable sequence of 1's and 2's, so that the expansion is to the base 3. Then, no matter how this is done, the boundary set will consist of .1 , .2 , .01 , .02 , .001 , .002 , ...

Nor does a countable iterate set imply that all the numbers on the boundary will be rational, as is seen by the following example:

λ'= .20212002121120002121121112000021211211121111200000......

(base 3)

Then the boundary will contain the number

m = .212112111211112111112111111 2 ...... , as well as all of its iterates, and limit points .2 , .12 , 112 , .1112 ,......0212 , .00212112 , .0002121121112 ..etc. The union of all these sets will still be countable.

Definition:

Let C^n designate the set of all reals $0 < \alpha < 1$, for which the closure of the iterate set, base n, is countable.

Theorem IV:

C^n is a group under addition (mod 1) : α , $\beta \varepsilon C^n$ implies $\gamma = \alpha + \beta$ (mod 1) ε C^n .

The result depends on the following lemma , (which is stronger than the theorem itself!) :

Lemma 1 :

If A and B are two countable sets of real numbers with the property that $\boxed{\overline{A}, \overline{B}}$ are both countable and compact, then $\boxed{\overline{A+B}}$ is also countable and compact. Here A+B is the "sum set" obtained by adding all elements of A to all elements of B.

Proof:

It is clear that A+B is countable, since this sum is simply a countable collection of countable sets. We show that, under the given conditions of countability and compactness, that $\boxed{\overline{A+B} = \overline{A} + \overline{B}}$

Let ψ be a limit point of A+B. There therefore exists an infinite set of numbers $\boxed{\gamma_i \in A+B,\ \gamma_i = \alpha_i + \beta_i, \alpha_i \in A, \beta_i \in B}$ **converging to ψ. Consider the two sets** $\boxed{\alpha = \{\alpha_i\}, \beta = \{\beta_i\}}$**. Assume first that the set β is finite. Then there must be at least one element β^* in b , and an infinite subset α' of α, such that a sequence of numbers of the form** $\boxed{\gamma_i' = \alpha_i' + \beta^*, \alpha_i' \in \alpha'}$ **converges to ψ. By compactness, there is a limit point θ of α', where θ is an element of A and $\theta + \beta^* = \psi$.**

Next suppose that both α and β are infinite sets. There must exist a subsequence of the sequence $\boxed{\gamma_i, \gamma_i^* = \alpha_i^* + \beta_i^*}$**, still converging to ψ, such that** $\boxed{\{\alpha_i^*\}}$ **converges to a unique limit point** $\boxed{\rho \in \overline{A}}$**. The** $\boxed{\beta_i^* = \gamma_i^* - \alpha_i^*}$ **must therefore also converge to a unique value** $\boxed{\mu \in \overline{B}}$

QED.

Proof of Theorem IV: Let $\gamma = \alpha + \beta$. The elements in the iterate set of g can be expressed as $\boxed{\gamma_k = (\alpha+\beta)n^k - [(\alpha+\beta)n^k]}$**. Then, either**

(i) $\boxed{[\alpha n^k] + [\beta n^k] = [(\alpha+\beta)n^k]}$**, or**

(ii) $\boxed{[\alpha n^k] + [\beta n^k] - 1 = [(\alpha+\beta)n^k]}$

Divide the collection $\{\gamma_k\}$ into two classes C_1 and C_2, to both of which we adjoin a 0 . where the elements of C_1 obey condition (i), those of C_2 obey condition (ii) . The elements of C_1 are sums of iterates of $\alpha + \beta$, therefore $\boxed{C_1 \subset A+B}$**. By the lemma** $\boxed{\overline{C_1}}$ **is countable.** $\boxed{\overline{C_2}}$ **is also countable, because the effect of the -1 is to shift the elements of C_2 into the interval (0,1) , hence** $\boxed{C_2 \subset A+B-\{1\}}$**, the expression representing, as usual, term by term set addition.**

Invoking the lemma once again, it follows that the set $\boxed{\overline{C_1 + C_2}}$ **is countable. Because of the zero element in each set, this includes the set** $\boxed{\overline{C_1 \cup C_2}}$**. The theorem follows.**

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DefinitionS :

By E_n we shall mean the set of all real numbers (mod 1) whose representation to the base n is either patterned or effectively patterned.

By J_n we shall mean the set of all reals (mod 1) which generate an iterate set with countable closure. There are several interesting theorems that relate E_n to J_n :

Theorem V:

If ρ is an irrational patterned n-cimal, then $\boxed{\overline{I_n^\rho}}$ is a perfect set, and conversely.

Proof:

Since ρ is patterned , it and everyone of its iterates will be limit points of $\boxed{\overline{I_n^\rho}}$. Therefore all the points of $\boxed{\overline{I_n^\rho}}$ are limit points and, since it is compact, it is perfect. Conversely, if $\boxed{\overline{I_n^\rho}}$ is perfect then every point is a limit point, and ρ is a patterned n-cimal.

Corollary :

If ρ is irrational and effectively patterned, then $\boxed{\overline{I_n^\rho}}$ is uncountable . All perfect sets are uncountable. Otherwise stated, ρ is not an element of J_n, and in fact $\boxed{E_n \cap J_n = Q}$, the set of rational numbers.

Observation: **The corollary is not true if we send n to ∞ . We are then dealing with the space S^I of all sequences of non-negative integers. If K = {k_α } is a set of elements of this space, then one says that a sequence λ is a limit point of K under the following condition: One can form an ordered subsequence of K , $\boxed{K_\lambda = k_{\alpha_1}, k_{\alpha_2}, \ldots.. k_{\alpha_j}, \ldots..}$ such that, given any integer N >0, the head block of λ o f length N appears as the head-block of all but a finite number of members of K_λ . This definition is an obvious extension of the identical property for n-cimals of finite base n.**

Let ϕ (h) = m = the highest exponent of 2 in h , and form the sequence $\Lambda = \phi(1)\ \phi(2)\ \phi(3)\ \phi(4) \ldots = 0102010301020104\ldots\ldots$

Then the closure of the iterate set of L in S^I coincides with the iterate set itself, which is obviously countable.

Theorem VI:

If $\boxed{\overline{I_n^{\rho}}}$ is countable, then all of its patterned n-cimals are rationals.

Proof:

By Theorem III every iterate set closure $\boxed{\overline{I_n^{\rho}}}$ has at least one patterned n-cimal. These cannot be irrational, since by the above theorems the iterate set closure would be uncountable.

We now have one way of characterizing the elements of J_n : Let γ be an element of J_n, $\boxed{\overline{I_n^{\gamma}}}$ its iterate set closure, and θ a patterned n-cimal in $\boxed{\overline{I_n^{\gamma}}}$. Then θ is a rational number.

Theorem VII:

If ρ is a patterned n-cimal, and r any rational number, then $\rho + r$ is effectively patterned, base n .

Proof:

Let $\gamma = \rho + r$. We will say that (ρ , r) is a *cascade* (mod n) if, in the addition process modulo n, a "1" is carried down from infinity to some finite position. In base n notation, if

$$\rho = \boxed{{}_n\varepsilon_1^{\rho}\ {}_n\varepsilon_2^{\rho}\ {}_n\varepsilon_3^{\rho}\cdots{}_n\,\varepsilon_k^{\rho}\cdots}$$

$$r = \boxed{{}_n\varepsilon_1^{r}\ {}_n\varepsilon_2^{r}\ {}_n\varepsilon_3^{r}\cdots({}_n\,\varepsilon_h^{r}\cdots{}_n\,\varepsilon_j^{r})\cdots}$$

, (where the segment in parenthesis is repeated periodically) , and

$$\gamma = \boxed{{}_n\varepsilon_1^{\gamma}\ {}_n\varepsilon_2^{\gamma}\ {}_n\varepsilon_3^{\gamma}\cdots{}_n\,\varepsilon_k^{\gamma}\cdots}$$

, then (ρ , r) is a cascading pair if there is an K>0 , such that for all k >K , we have :

$$\boxed{{}_n\varepsilon_k^{\gamma} = {}_n\varepsilon_k^{\rho} + {}_n\varepsilon_k^{r} + 1}.$$

Suppose ρ patterned, r rational and (ρ,r) non-cascading. Let the period (after some point) of the representation of r (base n) be p . Let $\boxed{L_{0,p+1}^{\rho}}$ designate the block of the first p+1 digits in ρ . Since ρ is patterned there must be infinitely many copies of this block scattered throughout ρ . Call the collection of all these blocks D (L , p+1) . An infinite number of these blocks must have an initial point q_k such that the corresponding entry in r , namely $\boxed{{}_n\varepsilon_{q_k}^{r}}$, will always be the same digit relative to the block $\boxed{L_{0,p+1}^{\rho}}$ which is being repeated periodically. Let us fix this digit, say "d", and let W_d be the subset of the collection D(L, p+1) with this property.

Since (ρ ,r) does not cascade, there exists an infinity of locations l_j , for which $\boxed{{}_n\varepsilon_{l_j}^{\gamma} = {}_n\varepsilon_{l}^{\rho} + {}_n\varepsilon_{l_j}^{r}}$. This in fact implies the stronger property that , provided ρ is irrational, (the theorem otherwise reducing to a triviality), there are infinitely many locations $l_{j'}$ for which $\boxed{{}_n\varepsilon_{l_{j'}}^{\gamma} = {}_n\varepsilon_{l_{j'}}^{\rho} + {}_n\varepsilon_{l_{j'}}^{r}}$

Extend the first block of W_d to the first of these locations $l_{j'}$. Designate this extended first block as L_1 .The sum of L_1 with the corresponding block in r will contain no "1" carried into it from the rest of γ . There are therefore infinitely many blocks in W_d which can be extended to blocks identical in content to L_1 and which, when added to r will produce identical blocks in γ in which there is no 1 carried down from the remainder of γ .

Therefore, in the case of a non-cascading pair (ρ ,r) , γ is effectively patterned.

Now suppose that (ρ , r) *is* cascading . Then , since a "1" is added systematically to every pair-wise sum beyond a certain point, the contents of

all blocks will be systematically altered in the same way. Therefore γ will be effectively patterned in this case as well. QED.

We are now in a position to characterize J_n more precisely :

Theorem VIII:

Let ρ be any positive real number. Every finite block , $L_{j,k}{}^{\rho}$ of entries in ρ which is repeated infinitely often in the n-cimal representation of ρ , will appear as the head block of at least one of the effectively patterned n-cimals in $\boxed{\overline{I_n^{\rho}}}$

Proof:

Assume that the block B, of length m , occurs infinitely often in ρ . Designate the set of all iterates of ρ which have B as their head block. Its closure $\boxed{\overline{T} \subset \overline{I_n^{\rho}}}$ consists of precisely those elements with B as their head block. Using arguments resembling those of Theorem III , we will show that the closure of T contains at least one effectively patterned n-cimal.

Case I : Suppose that for all elements of $\boxed{\overline{\overline{T}}}$, the block B occurs infinitely often. Then choose γ_1 on the boundary of T and form the set T' of all iterates of γ_1 which have B as their head block. Then clearly $\boxed{\overline{T'} \subset \overline{T},\ \overline{T'} \cap T = \varnothing}$. Proceeding in like manner one forms T'', T'''... etc. , as well as their closures As in Theorem III, the properties of compactness imply a non-vacuous intersection of their closures, implying the existence of at least one patterned n-cimal.

Case II : There exist elements μ in $\boxed{\overline{\overline{T}}}$ in which B only occurs a finite number of times. There must therefore be elements in the iterate set of μ which have B as their head block and nowhere else in their n-cimal representation.

Since r has infinitely many copies of B in its n-cimal representation, there must therefore be infinitely many iterates of ρ with B occurring only as their head block and nowhere else. Let G be the set of these. G is closed. If G' is its set of limit points and contains no effectively patterned n-cimals, then

$\boxed{\overline{G'} \subset \overline{G}}$. If G' contains no effectively patterned n-cimals, then $\boxed{\overline{G'} \cap \overline{G} = \varnothing}$ Once again the process is iterated, forming G', G'', G'''. Take their closures. Again, by compactness, the intersection of this series is non-vacuous, and contains only effectively patterned n-cimals with head block B.

Corollary:

The elements of J_n have the following structure:

Every block B occurring an infinite number of times in their n-cimal representation either:

(i) Develops longer and longer strings of periodic repetitions of B or:

(ii) is made up of a combination of a head block H, and a part B, with longer and longer combinations of the form HB , HBB , HBBB ,.... occurring in the representation.

This is because the set of patterned n-cimals in the closure of J_n is the set of rationals Q.

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Patterned Sets, Sequences , n-cimals

We now let $\boxed{A = a_0, a_1, a_2, \ldots\ldots}$ be a sequence of non-negative integers.

Definitions:

A *section point set* S is a strictly increasing sequence of indices , starting from $s_0 = 0$, which, when applied to the sequence A, creates a collection C = { C_k } of finite segments of A , given by :

$$\boxed{C_k = a_{s_k}, a_{s_k+1}, \ldots\ldots a_{s_{k+1}-1}}$$

An *algorithm* L is a sequence of non-negative integers which index a sequence of elements of C by concatenation. Thus, if $\boxed{L = l_0, l_l, \ldots. l_k, \ldots.}$, then the derived sequence P = P(A,S, L) is constructed from the other sequences as

$\boxed{P = C_{l_0} \Delta C_{l_1} \Delta C_{l_2} \Delta \ldots\ldots}$, where the Δ symbol is a concatenation operator that strings them together.

Example:

A = 0,1,2,3,.....,k,.....

S = 0,1,3,6,10.15,....n(n+1)/2 , ...

L = 010210321043210.......

Then $C_0 = (0)$, $C_1 = (1,2)$, $C_2 = (3,4,5)$, $C_3 = (6,7,8,9)$.... The derived sequence is therefore

$P = C_0C_1C_0C_2C_1C_0C_3C_2C_1C_0C_4C_3C_2C_1C_0$........

= 0120345120678934512O..........

A is the associated sequence to P. Summarizing:

A is the *associated sequence*

P is the *derived sequence*

S is the *section point set*

C is the *segment set*

L is the *code or algorithm*

n-cimal and iterate set have already been defined, Two definitions of a patterned n-cimal have already been given:

(1) The representation to the base n of a real number α is said to be patterned if α is a limit point of its own iterate set.

(2) A sequence in general is said to be patterned if every finite segment is reproduced elsewhere in the sequence.

An effectively patterned sequence in one which is patterned after a certain index N, that is to say, which has a patterned iterate. Likewise for effective periodicity.

We now give a 3rd definition of patterning which, for n-cimals, is equivalent to (1) and (2) above:

(3) P is a patterned sequence if and only if it is the derived sequence from some associated sequence A , and section point S, in accordance with the following algorithm:

L = 0102010301020104010201030102010501020103.........

that is to say λ (n) = m where m is the exponent of 2 in n+1 .

We will now establish an important relationship between patterned sequences and patterned n-cimals. Let the n-cimal representations of the real numbers α and β be designated as $\boxed{\alpha_{[n]}, \beta_{[n]}}$. If we define sequence addition as A(+) B = { a_k+b_k } = D , then , in general $\boxed{(\alpha+\beta)_{[n]} \neq \alpha_{[n]}(+)\beta_{[n]}}$. We also set $\gamma = \alpha + \beta$ by ordinary real addition and C = $\gamma_{[n]}$. Then , in general C ≠ D . In fact, D may not even be an n-cimal since it may have entries larger than n-1 . However, we can prove the following:

Theorem IX :

If α β γ A B C D are defined as above, then if D = A(+)B is a patterned sequence, then C will be an effectively patterned n-cimal.

Proof :

Case I : $a_k + b_k < n$ for all k. Then C and D are identical and C is patterned because D is.

Case II : There are only finitely many integers $\boxed{k_1, k_2, \ldots, k_s}$ for which $a_k + b_k \geq n$. Then D and C will be identical after s iterates. Since every iterate of D is patterned, it follows that C is patterned after s, hence effectively patterned.

Case III : There are infinitely many indices k for which $a_k + b_k \leq n-2$. Then let q designate the index of the first element of D which is less than or equal to n-2 . A "1" which is carried down to d_q from the rest of the summation process for $\gamma = \alpha+\beta$ will stop at this index and not produce anything that will be carried down to that part of C from the c_q to the beginning. Designate this block as $C_0 = c_0c_1c_2 ....c_q$. The corresponding block in $D = d_0d_1....d_q$ can be designated D_0 . Since D is patterned D_0 will be repeated infinitely many times in D. Call these copies D' D'' D''', D'''' ..... These correspond, indexwise in turn, to blocks C' C'' C''' C'''' etc., in C. Since c_q prevents anything from being carried into C_0 , the final number in C' , C'' , etc., will also prevent anything from being carried into these blocks . Therefore these blocks C' C'' and so on, will be identical copies of C_0 . Since there are, by hypothesis , infinitely many indices q with $a_q + b_q \leq n-2$ we can make C0 as long as we wish, and therefore C will be patterned.

Case IV : There are only finitely many $d_k \leq n-2$

(i) There are only finitely many $d_k > n-1$. Then, after a certain point, every dk will equal n-1 . In the process of converting this into C by carrying, the infinite sequence of n-1's will turn into an infinite string of 0's (in the same way that, in base 10 , 0.9999999999 = 1.000000000000) , C will then represent a rational number, and is therefore certainly effectively periodic and effectively patterned.

(ii) There are infinitely many $d_k > n-1$. Then $n-1 < c_k \leq 2n-2$ for infinitely many c_k . In the process of converting D into C , one therefore carries a "1" from infinity down to some index q . Thus the effect on each $d_k = a_k + b_k$ between d_q and infinity is to produce a corresponding element $c_k = a_k + b_k + 1$ (modulo n) . Since D is patterned, and since the transformation $c_k = d_k +1$ (modulo n) is a function of the value of d_k and not of its index, it follows that C will be patterned from the index q onwards. Thus C is effectively patterned.

This completes the proof.

Corollary:

If A(+)B(+)C(+)....(+) M = N is a patterned sequence, then the corresponding real number $\alpha + \beta+\gamma +...+ \mu = \nu$ will be an effectively patterned n-cimal. Proof by induction the above result.

Theorem X:

If $\boxed{\alpha_{[n]}}$ is effectively patterned, then

(A) : $\boxed{(\alpha + r)_{[n]}}$ is effectively patterned, where r is rational

(B): $\boxed{(k\alpha)_{[n]}}$ is effectively patterned, where k is any integer

(C): $\boxed{(\alpha/m)_{[n]}}$ is effectively patterned, m integer

(D) $\boxed{(r_1\alpha + r_2)_{[n]}}$ is effectively patterned , r_1 , r_2 rational.

(B) is a direct application of the corollary. (A) follows from (B) and (C) , because if r = s/t , then (tα +s) /t = α + r . Similarly, (D) follows from the combination of (A) (B) and (C) . Therefore we need only prove (C).

Proof of (C) : **Consider the process of dividing α by some integer m Let H(s) be the head-block of length s in the n-cimal representation of α . Since $\alpha_{[n]}$ is patterned it contains infinitely many copies of H(s) .**

In the division process one brings down each occurence of H(s) preceded by a certain remainder left over from the division made just before that occurence. This remainder must be less than m . Thus there are only finitely many possible remainders, and at least one of them r_1 , must occur infinitely ofter. Therefore the expression r_1H must be a dividend infinitely often. This means that the block in $\boxed{(\alpha/m)_{[n]}}$ corresponding to r_1H must occur infinitely often.

The same reasoning is then applied to the block HKH, where K is the filler block between two occurences of H in $\boxed{(\alpha)_{[n]}}$. This also appears infinitely often, and when divided by m is preceded by a remainder, at least one of which, say r_2 , must occur infinitely often. In the same way, there will be a remainder r_3 preceding the division of the block (HKH)L(HKH), L being, again, the filler between the occurences of HKH .

One thus develops a sequence of remainders $\boxed{r_1, r_2, r_3, \ldots\ldots}$. Since they are all less than m, there must be at least one of them, label it ζ , which occurs infinitely often in this sequence.

It follows that $\boxed{(\alpha/m)_{[n]}}$ will be patterned from the point at which the division process first produces the remainder ζ. Hence it is effectively patterned.

This completes the proof of Theorem X.

Corollary:

The property of being effectively patterned is independent of the base n. The proof follows the same lines as the above.

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Given real numbers α , β , both of which are patterned or effectively patterned, it doesn't follow that $\gamma = \alpha+\beta$ will be either patterned or effectively patterned. To investigate those situations under which the sum will produced another effectively patterned n-cimal, we define the *pattern set* , M .

Definition:

Let $\boxed{\alpha_{[n]} = \varepsilon_1^{\alpha}\varepsilon_2^{\alpha}.....\varepsilon_m^{\alpha}.....}$. The set of integers M = {M_k } is called the pattern set for $\boxed{(\alpha)_{[n]}}$ if

$$\boxed{\begin{array}{l} 0 = M_0 < M_1 <\ldots M_j <\ldots \\ \varepsilon_j^{\alpha} = \varepsilon_{j+M_k}^{\alpha} ,1 \le j \le M_{k-1} \end{array}}$$

The pattern set creates the division of the sequence $\boxed{\alpha_{[n]}}$ into the collection of segments C = {C_k } . It is not unique. Indeed, any infinite subsequence of a pattern set is also a pattern set!

Definition:

By $(M)^n$ we shall mean the collection of all n-cimals which have M as a pattern set . If M forms a pattern set for $\boxed{(\alpha)_{[n]}}$, then we will say that α *belongs to* $(M)^n$. When the base is obvious from the context, we will say, simply, that α belongs to M .

Theorem XI:

If α, β both belong to M, then the sum g = a + b , will be effectively patterned. For then the sequence $\boxed{(\alpha)_{[n]}(+)(\beta)_{[n]}}$ will have the same pattern set and, by Theorem IX , γ will be effectively patterned.

Corollary:

If α , β, γ , ..., ω , all belong to the same pattern set, then the number $\boxed{h = r_0 + r_1\alpha + r_2\beta + \ldots . r_k\omega}$, where the r's are rationals, is effectively patterned.

This corollary is not valid in the infinite case. Indeed, it is possible for an infinite set of patterned n-cimals to have the property that every finite subset of them all belong to the same pattern set, without there being any pattern set for the entire collection, an infinite set of inclusions with vacuous intersection.

One can also easily construct examples of 3 patterned n-cimals, any two of which share a certain pattern set, without there being a common pattern set for all three of them. For example,

$$\boxed{\begin{array}{l} \alpha_1 : 0, M_1, M_3, M_5, M_7, M_9 \ldots \\ \alpha_2 : 0, M_1, M_2, M_5, M_6, M_9, M_{10}, \ldots \\ \alpha_3 : 0, M_2, M_3, M_6, M_7, \ldots \end{array}}$$

It is assumed that these are the "minimal" patterned sets to which these numbers belong.

However, one has this nice theorem:

Theorem XII:

Let K be an infinite set of patterned n-cimals, all of which belong to the same pattern set M. Then all the members of the closure $\boxed{\overline{K}}$ belong to M. The theorem is all but self-evident, merely making the observation that there are no special ambiguities between infinite strings of 0's and infinite strings of the integer n-1 , owing to the repetitive character of the pattern algorithm.

Corollary:

The set of real numbers $(M)^n$ is a perfect set. By Theorem XII $(M)^n$ is closed. Also, every element ρ of (M)n can clearly be obtained as the limit point of other elements via, for example, a 'diagonalization' on ρ, which alters a single digit of ρ (and its reoccurences relative to the pattern set)) at a time.

Let P be a patterned sequence. If P has the property of having a section point set which is an arithmetic progression, so that all segments C_k of new material have the same length, then we will say that P is metric.

Theorem XIII:

Let α, β be metric patterned n-cimals, with generic segment lengths m_α, m_β. Then g = a + b is an effectively patterned n-cimal.

We will prove this for general metric sequences, then appeal to the fundamental theorem IX .

Since all segments of the patterning have equal length, the head segment $\boxed{C_0^\alpha}$ will reoccure at locations 0 , $2m_\alpha$, $4m_\alpha$, ... Likewise for the head block of β . It follows that the head segments of both a and b will initiate at locations $2Nm_\alpha m_\beta$ v, where N is any positive integer.

Likewise one can start with headblocks

$$\boxed{C_0^\alpha = 2^k m_\alpha , C_0^\beta = 2^k m_\beta}$$

By virtue of the pattern algorithm, these blocks will both reoccur at locations $\boxed{2^{k+1} Nm_\alpha , 2^{k+1} Nm_\beta}$ respectively. Therefore both blocks are initiated together at locations $\boxed{C_0^\alpha = 2^{k+1} m_\alpha m_\beta}$.

This shows that $\boxed{\delta = \alpha_{[n]}(+)\beta_{[n]}}$ is patterned. Therefore $\boxed{\gamma_{[n]} = (\alpha + \beta)_{[n]}}$ is effecctively patterned.

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We will be employing an extension of the non-negative integers, $Z_{(n)}^{ext}$. This consists of all non-negative integers represented in the base n , as well as all combinations of integers with an initial block of 0's . Thus, the following elements are in $Z_{(10)}^{ext}$. 23; 0 ; 01 ; 00002560643 , etc...... In contexts in which there is no ambiguity, we may refer to the elements of $Z_{(n)}^{ext}$ simply as "integers" . Otherwise they will be called 'extended integers' . The expression " the (extended) integer N is missing from α ", means that N occurs nowhere in its n-cimals sequence.

Example: Let ρ be a number which, expressed in base 10, lacks the integer "001" , and which includes every integer which does not have 001 as a subsequence. One way of doing this is to line up all integers which don't end in 0 and which don't have the combination 001 , and string them all together.

If N is absent from ρ , then all integers of the form HNK , where H and K are arbitrary, will also be absent from ρ . One can therefore say that N generates a set of integers all missing from ρ .

Likewise, if the integer N is present initially in ρ , then disappears, we can say that N is effectively absent from ρ . Let H_ρ stand for the class of all integers absent from ρ , and H_ρ^{eff} the class of all integers effective absent from ρ . We are interested in identifying the generators of these sets, absent integers not generated from any of their subsegments which are also absent from ρ .

There is a simple method for producing these generators. As all the elements of H_ρ and H_ρ^{eff} are of finite length, there must exist a minimum length L_0 for all tbe elements of H_ρ and a minimum length J_0 for the elements of H_ρ^{eff} . The elements of minimal length must be generators. So we remove from these sets all elements that contain them.
This leaves us with sets $H_{\rho,1}$ and $H_\rho^{eff}{}_{,1}'$. These in turn have elements of minimal length which generate sets that can be deleted. This method, which bears some relation to the sieve of Erastothenes, can be continued until all of

the generators have been identified. Since H_ρ^{eff} includes H_ρ the set of generators of the former includes that of the latter.

The generators of H_ρ will be called the *list* , designated L_ρ . Likewise the generators of will be called the effective list, L_ρ^{eff} .

Suppose first that L_ρ^{eff} is finite. Since L_ρ is included in L_ρ^{eff}, it follows that L_ρ is also finite. The converse also turns out to be true, although the proof is far from trivial:

Theorem XIII :

If L_ρ is finite , then so is L_ρ^{eff} .

Proof :

Assume L_ρ finite , and that the integer K is in L_ρ^{eff} but not in L_ρ . Then K occurs in the n-cimal representation of ρ only a finite number of times, l j . The structure of ρ therefore looks like this

$$\rho_{[n]} = XKYKZKW........$$

For the sake of the argument, lets say that ZK was the final occurence of K . We form an associated sequence I, by repeating ZK infinitely often:

$$I = XKYKZKZKZKZKZKZKZK.......$$

None of the subsequences of KZK are members of L_ρ , yet any subsequence S of I which is also a generator of L_ρ , must include KZK as its subsequence . That is S ε I , S a generator of L_ρ implies KZK ε S . Therefore S must include K . Therefore there must be an element of L_ρ which contains at least 2 copies of K .

Since the number of elements in L_ρ is finite, and since for any integer in L_ρ^{eff} there is a member of L_ρ with two copies of that integer, it follows that the number of elements in L_ρ^{eff} is also finite.

We next look at the case in which both L_ρ^{eff} and L_ρ are infinite . We distinguish two situations:

(i) $L_\rho^{eff} - L_\rho$ (= $\boxed{L_\rho^{eff} \cap (L_\rho)^C}$) is finite

(ii) $L_\rho^{eff} - L_\rho$ is infinite

We postpone discussion of these cases until after a number of theorems relating n-cimals with a finite list to patterned n-cimals have been presented. Recall that the 'list' of the n-cimal representation of a real number ρ is the collection of integers absent from this representation, and a patterened n-cimal is a real number that is also a limit point of the iterate set of its n-cimal representation.

Definition:

The collection of all real numbers such that the list of its n-cimal representations is vacuous will be designated as $\boxed{\Omega_n}$

Theorem XIV :

Every element of $\boxed{\Omega_n}$ **is patterned.**

***Proof*:**

Let H be a head segment of α of arbitrary length. Then if N is any integer, the concatenated integer NH will be found somewhere in α . This is enough to show that α is patterned.

Theorem XV:

Let $\boxed{\mu(\overline{I_n^{\alpha}})}$ **designate the Lebesgue measure of the closure of the iterate set of a . Then μ = 1 if and only if** $\boxed{\alpha \in \Omega_n}$ **. This is a restatement of Theorem I.**

Theorem XVI:

Let L_α be the list for α (modulo n) . If Γ , the set of generators of L_α is has a finite number of elements, then α is effectively patterned.

Proof:

Denumbrate the members of Γ as $N_1 < N_2 .... < N_j$. (Since these integers can contain an initial string of 0's, the ordering is alphabetical) . Let m be the maximum length of an integer in Γ , and let M > m . Partition the n-cimal representation of α into blocks of length M .

$$\boxed{\begin{array}{l}\alpha_{[n]} = B_1 B_1 B_1 \ldots\ldots\ldots; \\ length(B_i) = M\end{array}}$$

The number of blocks with distinct content cannot exceed n^M . Therefore there is at least one block B* which is repeated an infinite number of times in this representation. We therefore consider the first iterate that begins with this block:

$$\boxed{\alpha^* = B^*........}$$

Since B* is repeated infinitely many times, this iterate has a structure which may be represented as B*AB*CB*D........

Designate as Q the sequence B*AB*A . We will find this form in α*. The reasoning is as follows: no subsequence of length m or less will be a member of L_α , hence it contains no generators of L_α .

Continuing this process, consider the form B*AB*CB*AB*C. This, too, has no generators of the list, and therefore this form will be repeated in α* . Proceeding in this fashion, we see that there are arbitrarily long segments of α* which reoccur. Therefore α is effectively patterned.

Theorem XVII :

If a and b are reall numbers and $\boxed{L_\alpha^{eff} = L_\beta^{eff}}$, then their iterate sets have identical boundaries : $\boxed{B_\alpha = \overline{I_n^\alpha} - I_n^\alpha = \overline{I_n^\beta} - I_n^\beta = B_\beta}$, anc conversely.

Proof:

If any of the elements of Z^{ext} is effectively absent from both α and β , then it will , in the long run, be eliminated ftom their iterates,and not appear as a subblock of any of their limit points.

Conversely, suppose $B_\alpha = B_\beta$, and form the set: $\boxed{S = \bigcap_{\rho \varepsilon B_\alpha} L_\rho}$, that is to say, the intersection of all lists (*not* effective lists) of all the elements of B_α (= B_β) . S will then be the effective list for both α and β . [1]

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[1] **In order that the theorem be strictly true we must specify that a number like.34000000000 is not written as . 3399999999, since these have different effective lists (3 ultimately dissapears from the first sequence, whearas 3 and 4 ultimately disappear from the second.)**

By virtue of this theorem we are able to give a complete characterization of B_α :

Corollary:

B_α consists of precisely those reals ρ , whose list includes L_α^{eff} .

Proof:

Note that any integer not generated by L_α^{eff} must end up as the headblock of some limit point in B_α . This shows that the content of Ba depends only on that of L_α^{eff} .

The following lemma is of interest because it is a simple redefinition of the notion of a patterned sequence in terms of the properties of lists:

Lemma:

A sequence σ is patterned if and only if $L_\sigma = L_\sigma^{eff}$. Proof immediate from the definitions of these terms

Corollary:

If α is patterned, β has the property that $L_\alpha = L_\beta^{eff}$, then β is effectively patterned and $\boxed{\overline{I_n^\alpha} = \overline{I_n^{\beta_k}}}$ for some iterate β_k of β .

Proof:

Under the given hypothesis and from the lemma one has $\boxed{L_\alpha = L_\alpha^{eff} = L_\beta}$. Letting B once more be the set of limit points, we see that $\boxed{\overline{I_n^\alpha} = B_\alpha}$, since for a patterned n-cimal, the border set coincides with the closure of the iterate set. It therefore follows that $\boxed{\overline{I_n^\alpha} = B_\beta}$.

Our next step is to show that $\boxed{\overline{I_n^\beta}}$ has only finitely many isolated points. This implies the existence of a patterned iterate.

Let $\boxed{S = \overline{I_n^\beta} - B_\beta}$ be the collection of points that are not limit points, i.e. the isolated points. If the corollary is false S can't be finite, for then only a finite collection of iterates would fail to be limit points and β would be

effectively patterned. However, since $\boxed{\overline{I_n^{\alpha}}}$ is a perfect set, then B_β is also a perfect set. S must then be an infinite discrete set between 0 and 1 whose closure is a perfect set. which is impossible.

The corollary implies that an examination of the effective list of an number α will provide enough information to say whether or not α is effectively patterned. This translartes into another corollary:

Corollary:

The collection $J^{eff} = \{L^{eff}\}$ of all effective lists, that is to say, all sets of generators of extended integers that are eventually missing in some n-cimal expansion , can be decomposed into sub-collections $\boxed{J = J_N \cup J_P}$
For any set M in J_N , none of the numbers for which M is an effective list are either patterned or effectively patterned. For any set D in J_P , all of the numbers for which D is an effective list are effectively patterned (or patterned) .
"Patterning" thereby translates into a structural condition on effective lists. Our theorems show us that all finite effective lists are in J_P . All of the effective lists of J_N are therefore infinite.

Some properties of J_N and J_P : **Since effectively patterned n-cimals are determined by patterned lists, it follows that a real number α is effectively patterned if and only if the cardinality of $\boxed{L_\alpha^{eff} - L_\alpha}$ is finite. or there will be some member of the iterate set which is patterned, and for this element the list and the effective list coincide.**

Similarly, let β be some number which is not effectively patterned, base n, . Then $\boxed{L_\beta^{eff} - L_\beta}$ must have infinitely many members. This has several consequences. The members of J_P can be either lists or effective lists of real numbers, whearas the members of J_N cannot be lists of any real number.

This implies a natural decomposition of the collection , J, of all lists , into Jp and a set J* of all lists which cannot also be effective lists.

Summarizing:

(i) If $\boxed{L_\alpha^{eff} - L_\alpha}$ **is finite, then α is effectively patterned or patterned**

(ii) If $\boxed{L_\alpha^{eff} - L_\alpha}$ **is infinite, then α is neither effectively patterned nor patterned. In this case** $\boxed{L_\alpha^{eff} \subset J_N \ \& \ L_\alpha \subset J^*}$.

We establish an equivalence relation: $\boxed{\alpha \approx \beta \leftrightarrow \overline{I_n^\alpha} = \overline{I_n^\beta}}$ **. It is easy to check that this satisfies the 3 conditions of equivalence, Identity, Commutativity and Transitivity. We have that** $\boxed{\alpha \approx \beta \leftrightarrow L_\alpha^{eff} = L_\beta^{eff} = L_\alpha = L_\beta}$ **. Clearly any number is equivalent to any of its n-cimal iterates. If α is equivalent to ρ and ρ is not an iterate of α , then** $\boxed{\rho \in \overline{I_n^\alpha} - I_n^\alpha = B_\alpha}$**. It may also be the case that a is an iterate of r . If this is not the case then we have**

$$\boxed{\rho \in \overline{I_n^\alpha} - I_n^\alpha = B_\alpha \ \& \ \alpha \in \overline{I_n^\rho} - I_n^\rho = B_\rho}$$

Let Σ_α be the equivalence class of which α is a member. Clearly Σ_α is contained in $\boxed{\overline{I_n^\alpha}}$**, though it may be properly contained. It is a simple matter to construct the generators of Σ_α : they generate all of Σ_α by iteration, yet none of them iterate into the other . Although** $\boxed{\overline{I_n^\alpha}}$ **will usually contain a mixture of patterned and unpatterned elements, all of the elements of Σ_α are patterned, since α is patterned and they all have the same iterate set closure, which is a perfect set which, by a previous theorem is always generated by a patterned n-cimal. In fact we have shown that:**

Theorem XVIII:

(i) All the elements of Σ_α have the same list as α .

(ii) $\boxed{\overline{I_n^{\alpha}}}$ consists of all n-cimals whose list includes L_{α} .

(iii) The structure of the generator set $G(\Sigma_{\alpha})$ is very complicated.

Examples:

I. Let μ = 0.1010010001000010000001........ , base 2 . Clearly the sequence is not patterned. The list L_{μ} consists of the integers 11 , 0101, 001001 , etc. The effective list, in addition to all these elements of L_{μ} , contains every subsegment of m except "0" , and those with only a single 1 in them, such as 1, 010, 001 , 100, ... etc.

Clearly the effective list contains infinitely many integers beyond the list.

II. Still working in base 2 , consider the number:

τ = 0.1011010111010110101111010110101110101101011110

This is evidently patterned, the simplest example of a non-rational patterned binary. Its list , which coincides with its effective list, includes 00,101010, 11011, etc. Any binary 2-cimal whose list includes the elements of this list will have integer subsegments all of which occur in τ infinitely often.

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Definitions:

I. Let us assume that the n-cimal $\boxed{\alpha_{[n]}}$ is patterned. We will say that a is *initiating* , if there is no patterned n-cimal $\boxed{\beta_{[n]}}$ such that $\boxed{\alpha_{[n]}}$ is one of its iterates. Likewise we will say that is a k^{th} *iterate patterned n-cimal* if there exists an initiating β such that $\alpha = \beta_k$ (base n) .

Now it is possible that there exists a number β^k *for every k* , of which α is a k^{th} iterate; that is $\boxed{\alpha_{[n]} = (\beta^k_{[n]})_k}$. One can then say that $\boxed{\alpha_{[n]}}$ is infinitely back-extendible . Likewise, the phrases , " α is back extendible to k places " , or " α is a kth iterate" , are synonymous. One can also more informally speak of a "pattern iterate", without specifying the number of places between it and some initiating n-cimal .

II. Let α be a pattern iterate which is not infinitely back extendible. We will say that a is braided if there are at least two distinct initiating patterned n-cimals , β and γ , such that α is a patterned iterate of each of them. The set Γ of initiating n-cimals for α will be called its *braids* . Since a is not infinitely back-extendible the number of elements in Γ is finite, and may be called the 'braiding number' of α .

The choice of the term braid derives from the generic construction for forming a pair of braids. Let

$\mathbf{B} = B_1B_2B_3.......B_n.....$ be a sequence of integers, in the sense previously defined. The construction of a braided pair β , γ proceeds in stages:

Stage 1:

$\beta : B_0$

$\gamma : B_1$

Extend β in the manner of the pattern index function, and transpose the result down to γ :

Stage 2:

$\beta : B_0B_1B_0$

$\gamma : B_1B_1B_0$

Stage 3: Extend γ in a similar fashion, then adjoin the result to β :

$\beta : B_0B_2B_0\ (B_3(B_1B_2B_0)$

$\gamma : (B_1B_2B_0)B_3(B_1B_2B_0)$

Stage 4: Extend β in a similar fashion, then adjoin the result to γ :

$\beta : (B_0B_2B_0\ (B_3(B_1B_2B_0))B_4(B_0B_2B_0\ (B_3(B_1B_2B_0))$

$\gamma : (B_1B_2B_0)B_3(B_1B_2B_0)B_4(B_0B_2B_0\ (B_3(B_1B_2B_0))$

And so forth. Since the initial blocks are different these braids are distinct. The iteration number for β is the length of block B_0 , for γ it is the length of block B_1 . The sequence of indices after the intial block may be called the "braided patterned index function", or simply the "braiding function" and is given by:

N = 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18.......

y(N) = 2 0 3 1 4 0 2 0 5 1 2 0 3 1 6 0 2 ....

The algorithm for the braiding function will be presented without going into the details of the method of finite differences used to derive it:

Let M be any strictly monotonic sequence of members of Z^+, that is positive integers in the usual sense.For simplicity we require that 1 be in M. Let q be any positive integer. We decompose q over M as follows:

Let m_1 be the largest element of M which is less or equal to q. Set

$k_1 = [q/m_1]$

$q_1 = q - k_1m_1$.

Treating q_1 in the same fashion we determine m_2, the largest element in M which is less than or equal to q_1. Likewise:

$k_2 = [q/m_2]$

$q_2 = q - k_2m_2$. Continue in this fashion until stage j at which we find $q_j = 0$. We have thereby decomposed q as:

$$\boxed{q = k_1m_1 + k_2m_2 + \ldots k_{j-1}m_{j-1}}$$

We may apply this process to any set M, the squares, factorials, triangular numbers, etc., thereby forming a "basis representation" over M.

In particular, let M be the Fibonacci numbers indexed as follows:

$f_2 = 1$, $f_3 = 2$, $f_4 = 3$, ...$f_5 = 5$, $f_6 = 8$, ...

An expression for the braiding function ψ may be obtained by such a decomposition of its even arguments over the Fibonacci series.

If n is even, we compute ψ as follows:

(i) Decompose n/2 over the Fibonacci series in the manner indicated. Since $2f_k > f_{k+1}$ by construction , the coefficients (k_j) of the representation will all be zeroes or ones . Hence $\boxed{n/2 = f_{k_1} + f_{k_2} + \ldots f_{k_j}}$ uniquely. Then ψ(n) is given by the lowest index in the Fibonacci decomposition of n/2 .

(ii) When n is odd the process is simpler

If both n and $\psi(n-1)$ are odd , then $\psi(n) = 1$

If n is odd and $\psi(n-1)$ are even , then $\psi(n) = 0$

Comparing this with the first 17 places of the braided indexing sequence given above, one sees that this algorithm gives the correct results. It is also clear why the function begins with the argument "2" rather that "1" or "0" . Because of the generic character of the braiding process one sees that a pair of patterned braids may be derived from any associated function.

How does this relate to the boundary properties of iterate sets? If β and γ are braids of the same patterned n-cimal α , one has clearly:

$$\boxed{\begin{array}{l} I_n^{\beta} \supset I_n^{\alpha} \; ; I_n^{\gamma} \supset I_n^{\alpha} \\ \\ \overline{I_n^{\alpha}} = \overline{I_n^{\beta}} = \overline{I_n^{\gamma}} \end{array}}$$

It follows that both β and γ are limit points of I_α . If β and γ are both initiating patterned n-cimals , and equivalent in the sense of having identical iterate set closures, one can ask if they must be braids of some common iterate α ? One quickly sees that this is not necessarily the case: equivalence depends only upon the loose condition that they have the same list L .

A final theorem concerning the structure of Ωn , all of whose elements are infinitely back-extendible.

Theorem XIX :

$\mu (\Omega_n) = 1$

That is to say, almost all real numbers are, to any base n, patterned n-cimals with vacuous list.

Arrange the elements of Z^{ext} by alphabetical ordering as $N_1 , N_2 ,...N_k$. Let λ_k be some patterned n-cimal whose list contains only the integer N_k . Then the elements of $\boxed{\overline{I_{\lambda_k}}}$ will be precisely those real numbers

with lists that include N_k . By a basic theorem of ergodic theory, and also by the opening theorems of this paper, one has

$$\boxed{\begin{array}{l}
\mu(\overline{I_{\lambda_k}}) = 0 \\[2em]
\mu(\overline{\bigcup_{1}^{\infty} \overline{I_{\lambda_k}}}) = 0 \\[2em]
\Omega_n = I - \bigcup_{1}^{\infty} \overline{I_{\lambda_k}} \\
\therefore \mu(\Omega_n) = 1
\end{array}}$$

#31...

#32...