Projective Concepts & Projective Constructs in Relativity and Quantum Theory

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1. Preface

Many fields of contemporary physics are modeled by non-Euclidean geometry. [1,4,19,23,32] . [1] For most purposes, the 3 standard 2-D geometries suffice : Euclidean, elliptic and hyperbolic. It is pointed out in this paper that there are actually 9 such geometries, not counting variants arising from particular restrictions . Each of these non-Euclidean geometries may in its turn be embedded up to isomorphism into the real or complex projective planes, RP^2 and CP^2 . [1, 11]

Geometric constructions derivable from the properties of lines, pencils and points of pure projective geometry will be called "projective constructs". Many of the fundamental objects in modern physical theory are clearly projective constructs. For example, the fiber bundle of light cones over Minkowski Space is a projective construct.

For every projective construct there is a corresponding dual construct : one switches the terms "line" and "point" , "intersection" and "collineation", etc., in the sentences that describe them.

A postulate for the physics of projective constructs is stated part way through the paper:

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[1] Numbers in brackets refer to the Bibliography

Projective Postulate

" If the description of a natural observation is a projective construct, then its dual also exists in nature. "

The argument is mathematical, physical, and above all philosophical: the ontology of projective entities is such that they must be cast in the form of dual pairs to merit the existential predicate.

When the two principles of special relativity, the "Light Principle" (LP) , and "Relativity Principle" (RP) are dualized, the resultant model, dual-Minkowski space, bears a strong resemblance to the Hubble Field combined with the Big Bang. It is possible that the expansion field of the cosmos may come directly from the application of the projective postulate to special relativity.

The final section of this paper applies these ideas to quantum theory. Arguments are presented to show that "uncertainty" is an fundamental magnitude of nature. The fiber bundle of "uncertainty parabolas " at each point of time/moment space, J, forms the basis for wave-particle duality.

2. Summary

The standard axioms for the projective plane in 2-dimensions exclude parallel lines. *However, the concept of parallel lines belongs to projective geometry.* This because this concept can be formulated entirely in terms of the primitive notions of projective geometry: 'point' , 'line' , 'intersection', 'colinearity', 'betweenness' , etc. .

A concept stated in the language of projective geometry will be called a "projective concept " .

The existence of an entity P which can be described as a projective concept is equivalent to the existence of its dual , Q, obtained by switching the words 'line' and 'point', 'intersection' and 'colineation', in the defining statement for P. For example: the concept dual to *parallelism* may be called *seperalism* [2]: points A and B in a given geometry are *seperal* if there is no straight line on which they are collinear.

Projective constructs are models for projective concepts that can be embedded in the projective plane, with redefinitions of the notions of "line" and "point" based on the standard lines and points of the plane . This redefinition must be done in such a manner that "intersection" retains its customary meaning of 'set intersection' , and "colineation" retains its customary meaning of common membership in a line .

The first part of this paper describes all of the standard non-Euclidean geometries [12,...,17] , together with their seperal duals, as projective constructs derived from projective concepts. The second part applies this range of geometries to the representation spaces of modern theoretical physics. It is in connection with this model that the projective postulate is stated and applied.

In the final part of this paper we propose a specific non-Euclidean geometry , hyperbolic/dual-hyperbolic space , as the time/moment space of quantum measurement. What we call the 'knowable world' lies in the region below this parabolic fiber bundle and above the conic fiber bundle of special relativity. Events not in this space are either impossible or unknowable or both. [3]

3. Projective Concepts and Projective Constructs

[2] The unconventional spelling is deliberate; the word 'separate' is misspelt in standard English.

[3] There may however be reasons to assume their existence. See in particular Feynman [27]

There exist projective geometries RP^n (CP^n) at all dimensions. The situation in dimension 2 however has unique features which justify the application of the term *pure projective geometry* (PPG) :

(1) The dualism of line and point is perfect only in PPG .

(2) The higher projective geometries are built out of combinations of the properties of PPG .

(3) It is the ideal domain for embedding models of non-Euclidean geometries.

The virtues of PPG have been noted by Alfred North Whitehead in his thought provoking treatise on the subject, in which he calls it ' *The science of cross-classification* ' [5, pages 4,5]

There are standard models for PPG in both 2 and 3 dimensional manifolds :

(1) PPG can be modeled on the (2-dimensional) surface , S, of the sphere in 3-space, by identifying pairs of polar opposite points as "points", and defining as "lines" the great circles on S .

(2) An isomorphic model can be obtained through defining a "line" as a 2-dimensional linear sub-spaces, (planes) in Euclidean 3-space passing through the origin, and a "point " as the intersection sets of these planes, "Points" become members of the pencil of lines passing through the origin.

(3) The central projection of S onto a plane E^2 tangent to its south pole transfers the structure of PPG on S onto E^2. One is obliged to adopt the convention that the equator of S is sent out to a "line at infinity"

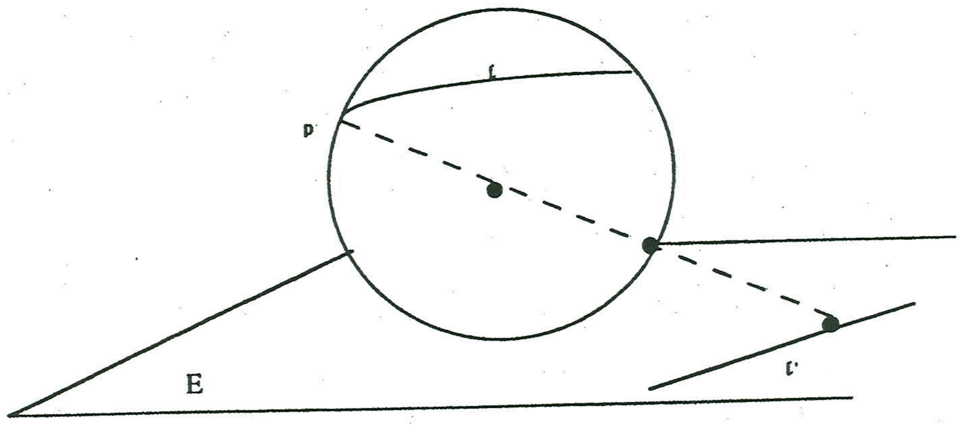


Figure 1

This way of speaking may be misleading since the 'line at infinity' is not a boundary in any sense. In fact it is indistinguishable from every other line on the plane. Although the construction may therefore seem a bit artificial , it is actually very useful in the practical applications of PPG that invoke limiting forms of constructions made in ordinary Euclidean geometry (Pappus' Theorem, Desargues' Theorem, etc.) [10].

The axioms of PPG were set down in the 19th century by von Staudt [5,9,10] , and appear in all standard texts . We assume that people who are interested in the ideas presented in this paper already have a fairly good idea of what projective geometry is. Much of the above can therefore be regarded as superfluous rhetoric, that is to say, 'rhetoric' in the colloquial sense.

Definition 1 : A *projective concept* shall mean any well-formed proposition couched in the language of projective geometry, set theory and the predicate calculus. That is to say, it employs no terms which cannot be expressed in terms of "line", "point", "intersection" , "colinearity" , "betweenness" and "membership" . These are subject to the following restrictions:

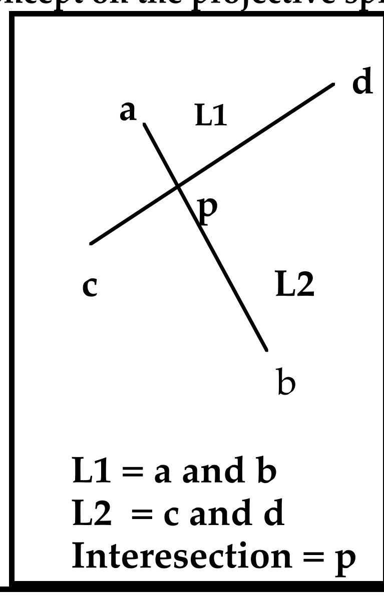
(1) An "intersection" defines a collection of points as some function of a collection of lines.

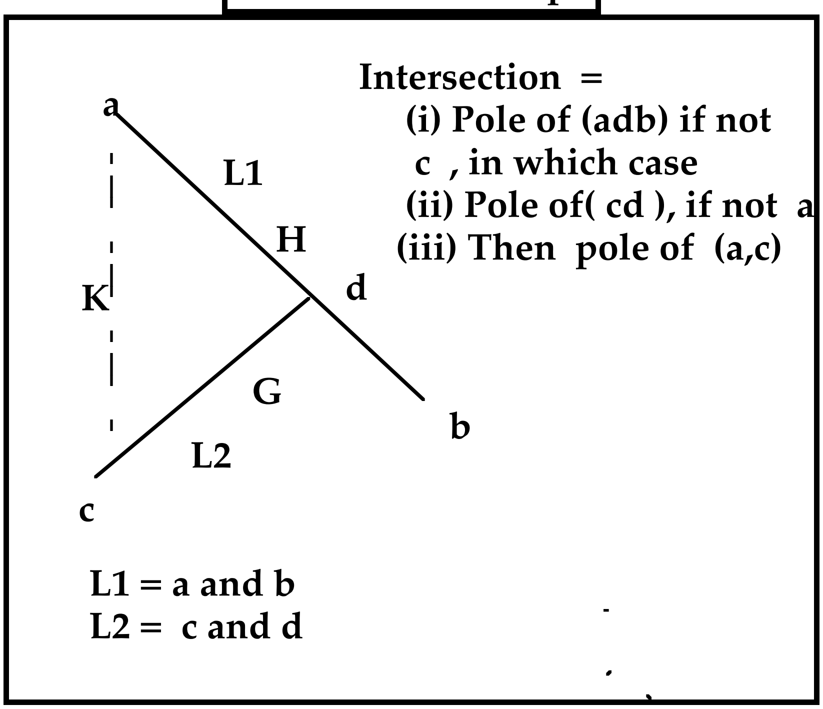
(2) A “colineation” defines a collection of lines as some function of a collection of points.

A non-trivial example of a projective concept:

Let L and M be two “lines” . Then the “intersection” of L and M will be a unique point p, which is never a member of either L or M

Here is a model for this concept on the projective sphere S :





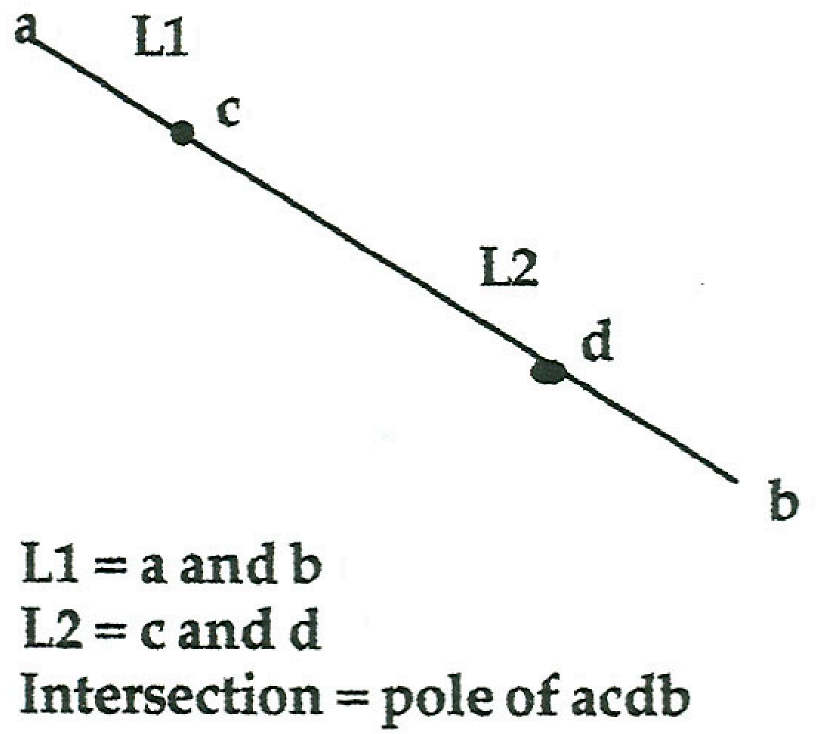


Figure 2

In this geometry , a 'line' is defined as *any non-polar pair of points* , (*with their corresponding polar opposites*) on S . Two such 'lines' 'intersect' in 3 possible ways:

(1) If no 3 points are collinear, (as defined by geodesics), and the segments formed by geodesics through (a,b) and (c,d) intersect in a point p, then p is defined as the "intersection" of the pairs (a,b) and (c,d)

(2) If d is collinear with the segment through (a,b) ,*then we take the pole of the geodesic, G, through (c,d)* as the "intersection", unless this coincides with a (or b) . In that case we take the pole of the geodesic H through (a,b) as the intersection. If this also coincides with c, then we draw a geodesic K through c and b and take its pole as the intersection. By definition, this cannot coincide with a (or b) .

(3) If the geodesics through (a,b) and (c,d) coincide in a geodesic H, then we take the poles of H as intersection .

In this way, every pair of lines, (themselves defined as pairs of

(polar pairs) of points), has a well-defined point associated with it which does not belong to either of them and may be called the intersection point. No “non-projective” ideas, such as “distance”, “neighborhood”, “finite” or “infinite”, etc., were used in this construction. It therefore qualifies as a model for a *projective concept* . It is not, however a *projective construct* in the sense that we will be using it in this paper :

Definition 2 : *A projective construct* is defined to be a model for some projective concept embedded in the pure projective plane. Specifically:

Let the projective concept be designated as □ . Let L□ and P□ stand for, respectively, the line-element model , and the point-element model, of □ . Then:

(1) L□ and P□ must themselves be projective concepts.

(2) L□ and P□ must be embeddable in PPG

(3) *In these constructions the words “intersection” and “colineation” (or their equivalents) occurring in □ must have the normal set-theoretic meanings of intersection (of lines considered as sets) and of membership (of points within in lines considered as sets) .*

Thus, for a projective concept to be considered a projective construct, we want geometry and standard set theory to come together. The above construction is not, therefore , a projective construct.

Clearly anything that can be “drawn” in PPG , using standard line and point concepts defined by its axioms , is a projective construct. Thus, the notions of “triangle” , “4th harmonic point” , “simple closed curve”, even “convex figure” , are projective concepts for which there exist projective constructs in PPG without alteration of the basic point and line elements. (One must enrich PPG sufficiently to allow for the definition of Jordan curves, but this presents no difficulty).

An example of a projective concept for which there can be no projective construct (by our definition) is a self-intersecting line . **Such a object implies two contrary uses of the word "intersection":**

(1) The line L "intersects " with itself in the sense of set theory. This intersection is just the entire line, L .

(2) The geometric "intersection" in the sense of the self-intersecting boundary of a curve . For a projective construct the two meanings must coincide. The only possible model for such a linear element , which is also a projective construct, is that in which every point p is also a line L . One might call this the "trivial geometry".

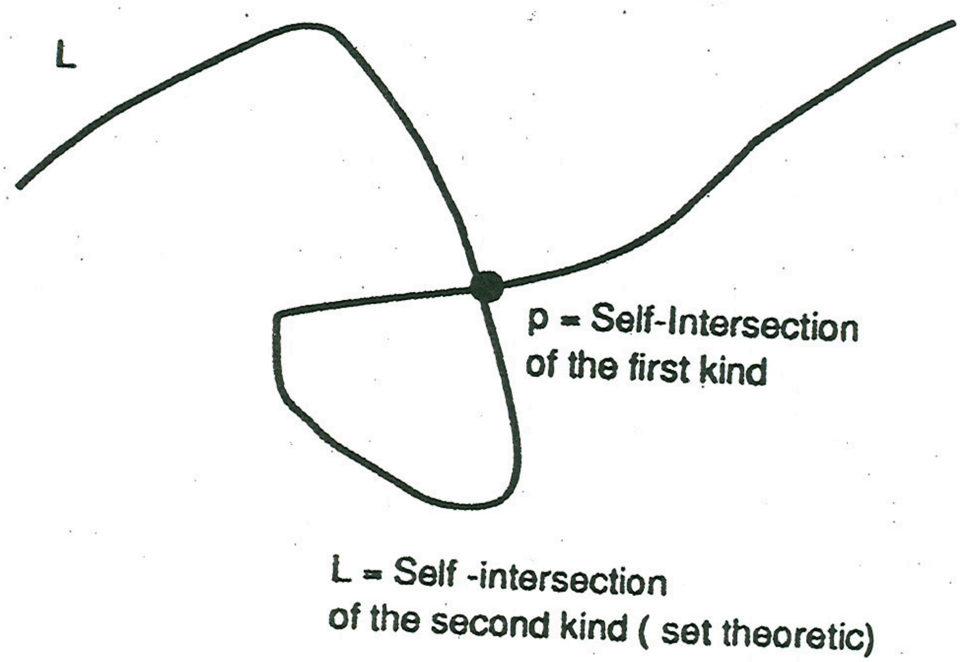


Figure 3

Here is the point of the exercise: Any statement □ in the form of a projective concept automatically generates its dual form , □ , which is also a projective concept. If there is a model for □ which qualifies as a projective construct, then there is automatically a model for □ which qualifies as a projective construct. If □ occurs in some physical theory and is correlated with an observable in nature, then there must be another observable in nature which correlates with □ : otherwise □ has been inconsistently defined, or the

projective space used to represent this aspect of the natural world is self-contradictory or inadequate . If we haven't found □ yet, we must keep looking for it. Otherwise we must abandon □.

As we will show, all of the standard, as well as some unconventional, non-Euclidean geometries can be modeled in PPG as projective constructs of projective concepts.

4. Point Bundle S_P and Line Bundle S_L

All non-Euclidean geometries involve relationships between a collection of points , S_P , and a covering, or fibration, consisting of the collection of pencils of distinguished sub-sets of S_P called lines, S_L . For the same of uniformity, these will both be called "bundles" .

In this paper we will have frequent need of making a careful distinction between:

(1) A line h, considered as a member of S_L

(2) The same line considered as a set of points from S_P

(3) A point p considered as a member of S_P , and

(1) The pencil of all lines $□_p$ intersecting in p.

Ontological difficulties may present themselves because it may happen , for certain constructions , that some of the elements of S_L will be vacuous. As an example, consider Euclidean geometry E^2 , modeled on a 2-sphere S, from which an equator H is deleted. This may be done in two ways:

(i) One deletes the entire point set of H from S

(ii) One deletes the line element H from the line bundle, but all the points on H are retained.

In the first case, all of the lines that cross the equator have a point missing (the "point at infinity") , producing the phenomenon of parallelism. In the second case, there are no parallels, but there is no line connecting pairs of points at infinity. That is to say, there is a 'forbidden direction' along the equator.

What are the corresponding procedures for the construction of E2 models for dual- Euclidean geometry E^{*2} ?

(1) One deletes the entire pencil of lines emanating from the North (South) pole ; or

(2) All these lines are retained ; one merely excises the North (South) pole from the geometry. One can then ask if the North/South pole "exists" in this geometry. Is it perhaps a "void point", like the "void sets" of conventional set theory? Dualism, indeed, seems to suggests that the concept of the "void point" merits a place in Projective Geometry.

The interdependence of point- and line bundles implies two distinct, yet co-extensive, ways of defining lines and points.

In the models we will construct for the non-Euclidean geometries, a line l will be defined as either:

(1) The set of points belonging to l ; or

(2) A particular member of S_L collinear with a certain pair of points

A point p , likewise, is either:

(3) A member of S_P ; or

(4) The intersection of two elements of S_L .

That this is more than mere quibbling can be seen from the following definition:

Definition 3 : Let $\square_L$ and $\square_P$ be , respectively, subsets of S_L and S_P , defined as projective concepts, for which there are projective constructs in PPG.

Let $\square_{int}$ be the set of all intersections of pairs of elements of $\square_L$; and $\square_{col}$ the collection of pre-images of the mapping of the elements of $\square_P$ into their (corresponding) lines in $\square_L$.(As observed above, these may sometimes be vacuous.)

Then any combination of the form ($\square_L$, $\square_P$) , ($\square_L$, $\square_{int}$) ,

(□col , □P), (□col , □int) , will be said to define a ***non-Euclidean geometry*** *, or simply a* ***geometry*** *, in PPG .*

The intent of this definition is best illustrated by an example: let C be a closed convex curve in the upper half-sphere $S^{1/2+}$. (This implies the construction of its polar opposite C* in the lower half sphere $S^{1/2-}$) . Remove □ = interior of (C,C*) from S, leaving us with a connected region □ = S/ □ on the sphere . In setting up a geometry over □ , one can choose for the elements of the line-space:

(1) The great circles that remain entirely in □ , or □L

(2) These, and in addition, all the truncated segments of great circles which touch on the boundary of □ . This is □ col .

These geometries are rather different.

(1) The geometry defined by (□L , □P) has many seperal point pairs, namely those that are not collinear with great circles entirely in □ . There are no parallels in this geometry, which is 'dual-hyperbolic' in our sense.

(2) The geometry defined by (□col , □P) has no seperal pairs; however, many 'lines' will not intersect. It is therefore ordinary hyperbolic geometry, indeed a Poincaré -Beltrami model [12,…17]

5. Classification and Construction of Non-Euclidean Geometries

The 3 traditional non-Euclidean geometries are distinguished through conditions on parallelism : one parallel (Euclidean) , no parallels (Riemannian) , and many parallels (Lobatchevskiian) , to a line L through a point p not on L . For convenience we will call these 'flat', 'elliptic' and 'hyperbolic' geometries.

Projective constructs could be used to model a far greater variety of non-Euclidean geometries. (***For example: define a "line" as any pair of lines in the projective plane, and a "point" as any 4-point set, no 3 of which are***

collinear. In this geometry, two lines define at most one point, and at most 3 lines pass through a point.)

Dualizing the 3 traditional geometries creates 3 geometries based on the properties of seperalism. Furthermore, there exist geometries which combine conditions on line parallelism and point seperalism in various ways.

The complete description of the projective concept of parallelism in PPG therefore involves 9 distinct geometries.

Definition 4 : **Two points in a geometric space are called** ***seperal*** **if there exists no line on which both of them are incident.**

Definition 5

(a): **A geometry will be said to be** ***dual-flat*** **, or have the** ***unique seperal property*** **if, given a point p and a line L on which p is not incident , there exists one and only one point q on L which, with p, forms a seperal pair.**

(b): **A geometry will be said to be** ***dual-hyperbolic*** **, or have the** ***many seperal property*** **if, given a point p and a line L on which p is not incident , there exist many points on L which form a seperal pair with p.**

(C): **A geometry will be said to be** ***dual-elliptic*** **, or have the** ***no seperal property*** **if any two points can be connected by at least one line.**

(6) Representations of Parallelism in PPG :

The non-Euclidean geometries

1. **Elliptic/Dual-Elliptic … No parallels ; No seperals (Riemannian)**
2. **Flat/Dual-Elliptic … Unique parallel; No seperals (Euclidean)**
3. **Hyperbolic/Dual-Elliptic … Many parallels; No seperals (Lobatchevskiian)**
4. **Elliptic/Dual-Flat … No parallels; Unique seperal**

(Dual Euclidean)

5. **Flat/Dual-Flat … Unique parallel; Unique seperal**

   (Two Dimensional Classical Space-Time)

6. **Hyperbolic/Dual-Flat … Many parallels; Unique seperal**

(Dual Minkowski)

7. **Elliptic/Dual-Hyperbolic … No parallels; Many seperals**

   (Dual Lobatchevskiian)

8. **Flat/Dual-Hyperbolic … Unique parallel; Many seperals**

   (Minkowskian)

9. **Hyperbolic/Dual-Hyperbolic ..Many parallels; Many seperals**

   (Quantum Time/Moment Space)

These geometries cover the combinations of parallel and seperal conditions . Most of them have found applications as representation spaces for models of phenomena in contemporary physical theories. One might of course also look at other species of Non-Euclid/Hilbert geometries for the modeling of phenomena : Non-Desarguen spaces , p-adic arithmetics , (which are Non-Archimedean) , and so on. For the present esoterica of interest mainly to mathematicians, they may well become useful in the future [10] . We now examine each geometry in turn:

(a) Elliptic/ Dual-Flat Geometries

Euclidean plane geometry is modeled directly in the projective plane by a familiar construction: Represent PPG as the geometry of great circles with identification of polar pairs on a spherical surface , S. Remove the equator H from the line bundle, and the equatorial points from the point bundle .

Set the plane, E^2 tangent to the sphere at the south pole. A Euclidean structure on E^2, or indeed any geometry, can then be mapped onto S by an *inverse projection* , the (*inverse central map*) from the center of the sphere bounded by S .

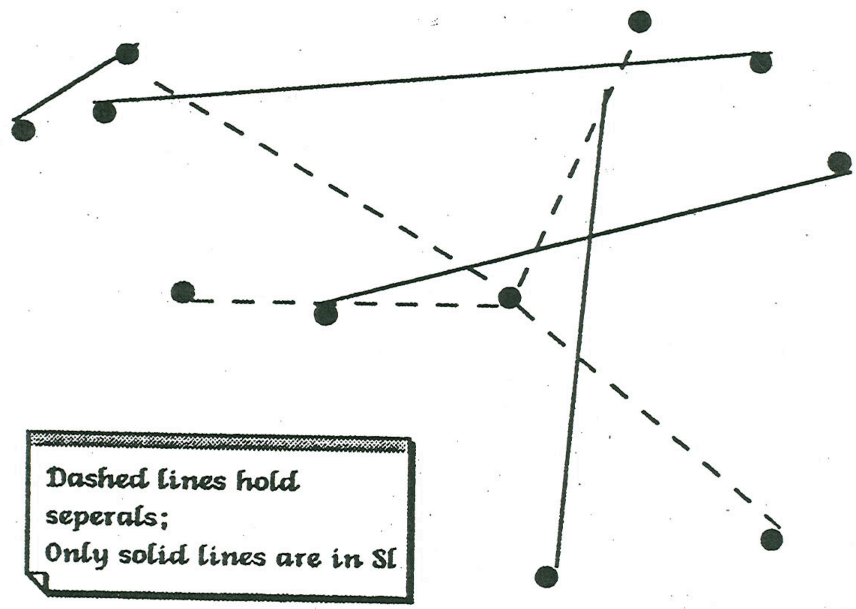


Figure 4

The upper half of S (minus H) is superfluous and can be discarded . Lines on the plane map into halves of great circles. Pairs of parallel lines become pairs of great circles intersecting at the equator, etc.

We seek the dual of this construction, on S and on the plane E^2. [4]

Dual to the removal of the equator is the removal of the North/South pole. Translating this onto the plane via the central map removes the and re-integrates the line at infinity into E^2 .

We also remove all lines in the line bundles $S_L (E^2)$, (and $S_L(S)$) passing through the North/South poles . This is dual-Euclidean geometry.

The basic properties of this geometry are:

(1) Any two lines intersect. Thus the geometry is *Elliptic*

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[4] E^2 will be used to refer, either to the Euclidean plane, or the unstructured Cartesian product R x R, of the real number line with itself.The specific meaning will always be clear from the context.

(2) Two points p and q on a longitude on S ,(or line passing through the excised origin on E^2) , will have no line through them.

(3) If H is a line not passing through the origin, and p a point not on H, then there is only one point q on H (recall that the line at infinity has been restored) which is seperal to p. Hence the geometry is *Dual Flat*

One can also represent elliptic/dual-flat geometry by a model which incorporates the line at infinity within the finite part of the plane. Remove the origin on E^2 , and parametrize locations by polar coordinates (□, □) , □ $\neq$ 0 . Define the line space as the set of curves:

$$\boxed{\begin{array}{l} \square = A\exp(b\square) \\ A > 0 \quad b \neq 0 \quad \square \quad < \square < + \end{array}}$$

In this geometry, two points with different values of □ but the same value of □ will be seperals. We will have occasion to return to this model in our discussion of Dual Relativity.

(b) Flat/Dual Flat Geometries

There are two simple ways of modeling flat/dual-flat geometry as a projective construct. Such a geometry will have both the "unique parallel" and the "unique seperal" properties. On S:

I. (i) Remove an equator from the line bundle.

(ii) Remove the points on the equator from the point bundle.

(iii) Remove the poles of this equator

(iv) Remove the line pencil that connects the poles,

The symmetry of this procedure insures a self-dual geometry

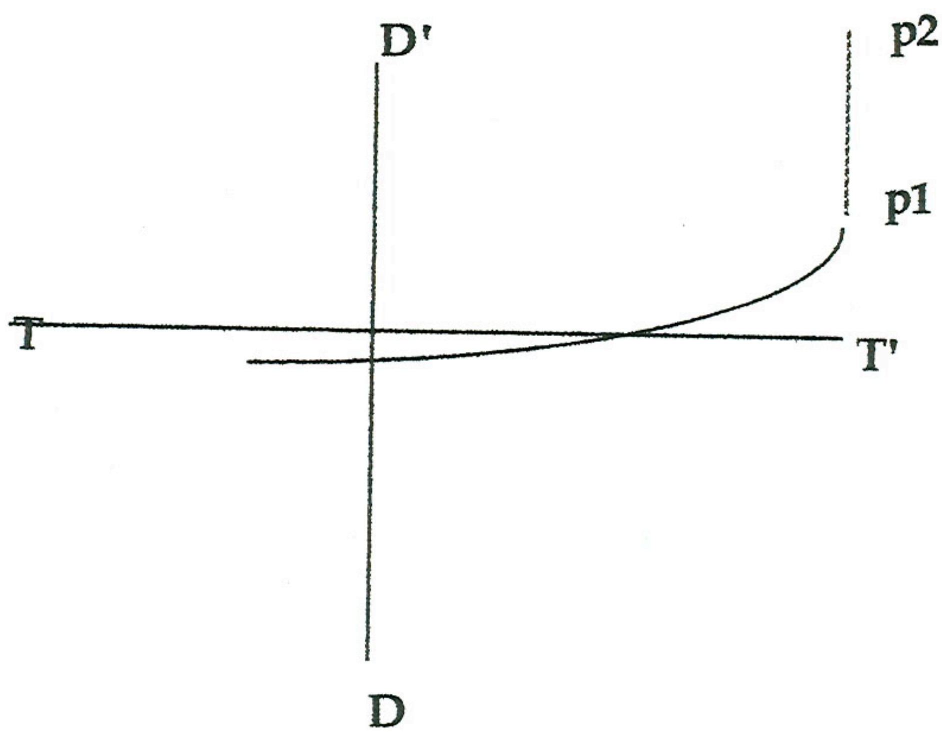
II. (i) Remove an equator from the line bundle , and the equatorial points from the point bundle .

(ii) Remove an entire pencil of lines between some polar pair *on the equator itself* .

Construction II creates a geometry essentially isomorphic to construction I. However, when the central map projects them onto the plane one obtains a very different pictures. In the second case, *a specific direction in the plane has been prohibited* .

This model is standard in classical physics. The trajectory of a particle , p, moving in a straight line can be represented using Cartesian coordinates , with abscissa "t" for time, and ordinate "x" for distance. A prohibition against vertical tangents on such a graph is equivalent to excluding the possibility of infinite velocities. Infinite velocities are ruled out in classical physics because two material particles cannot be in two different places at the same time.

In the following diagram , the points p_1 and p_2 on a vertical line are clearly seperal. The geometry produced on E^2 by construction I may be designated "polar", that of construction II, "equatorial" :



$T = f(D)$

$dT/dD \neq 0$

Figure 5

(c) Hyperbolic and Dual-Hyperbolic Geometries

Constructions for the hyperbolic geometries differ from those of flat and elliptic geometries, in that the mere removal of lines, line pencils, or points, from either the line bundle or the point bundle is inadequate to the task of embedding them in PPG . Poincare-Beltrami models and others require the introduction of closed convex curves , non-self-intersecting curves, (thus entirely in $S^{1/2+}$ ($S^{1/2-}$)) , for which " interior" and "exterior" can be defined. It is here that the distinction between a line defined as *an element of a collection of truncated lines* **, or as** *a certain kind of subset of the point bundle* **, becomes meaningful.**

Fix a line, H , as equator, with corresponding North/South pole □ = (N,S) . We assume that the notion of a simple , connected, non-self-intersecting curve □ = (C, C*) on S is clear. The interior of □ is defined as that region including □ .

□ is convex if:

(1) It does not intersect H

(2) Let p and q be interior points of □ . Then one of the two segments of the line (great circle) between p and q does not intersect □ .

Let □ be the exterior of □ , □ its interior . □ is certainly connected. Also, since □ does not intersect H , □ contains at least one great circle. □ does not contain any. Therefore we will call □ the "exterior geometry" and □ the "interior geometry" .

Theorem:

(1) There is a geometry inside □ , definable via a projective construct, which is hyperbolic/dual- elliptic , and a dual geometry inside □ which is elliptic/dual-hyperbolic.

(2) If □ is a polar pair of circles , and the radius of this circle is 1/4 of the arc of a great circle, then the interior geometry , and the exterior geometry are exactly dual.

Proof:

(1) Let a, b, c and d be any 4 points on □, arranged sequentially in either clockwise or counter-clockwise order. Connect a and d by a segment s_1 cutting across □. Connect b and c by a similar segment s_2 .

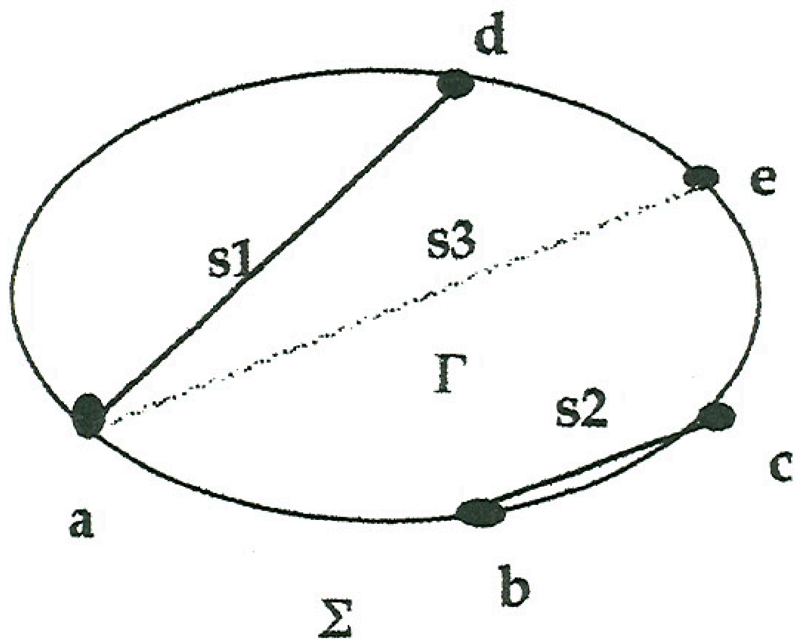


Figure 6

The Axioms of Order of Projective Geometry guarantee that s_1 and s_2 will not intersect. We can then consider the two segments as "parallels". If e is on the arc between d and c (not containing a or b) , then the line s_3 = (ae) intersects the line s_1 = (ad) only on the boundary. This allows us to treat □ as the Absolute in the Poincare-Beltrami construction for a hyperbolic geometry. Since C is convex, any two points in □ can be connected by a "line". This geometry is therefore also dual-elliptic.

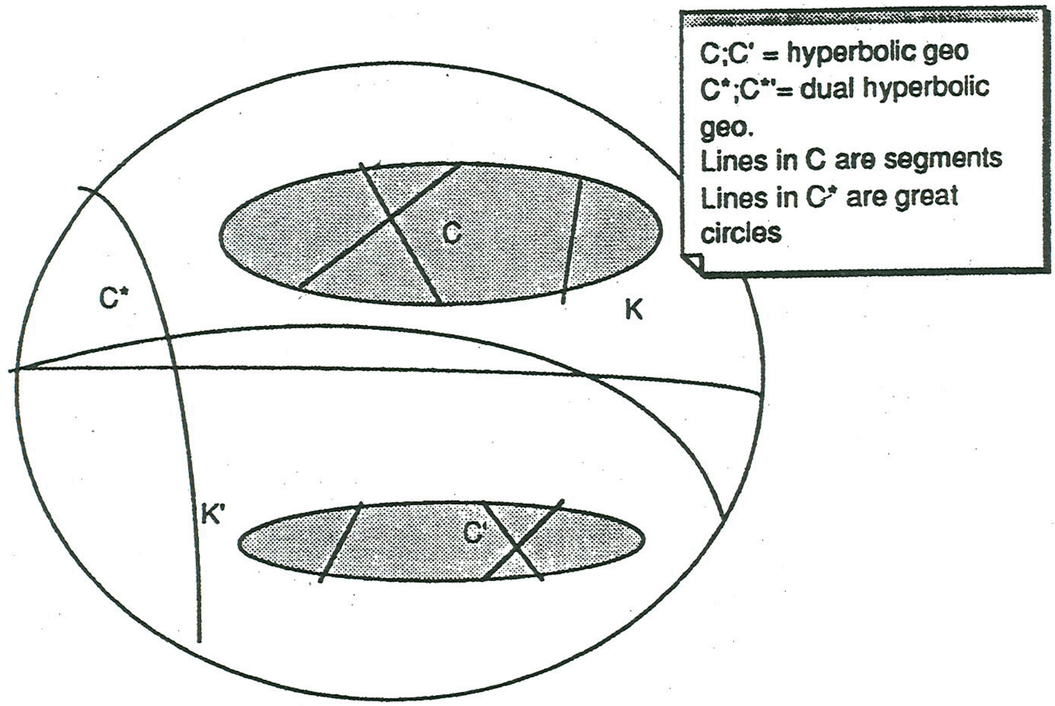


Figure 7

The geometry in the exterior region □, contains two kinds of lines: great circles and truncated segments of great circles.

(a) If we accept only the great circles in our line bundle , then the resultant geometry will have the following properties:

(i) If L is a line, p a point, then there are many seperals to p on L. Thus the geometry is dual-hyperbolic.

(ii) Any two lines (being great circles), will intersect. Thus the geometry is elliptic.

(b) If we agree to accept both great circles and truncated segments in the line bundle , then there are no seperal points. lines do not necessarily intersect and we have, once more, a hyperbolic/dual elliptic space.

(2) In the special case in which □ happens to be the circle that cuts all great circles between the North Pole and the Equator at their mid-point, then all of the great circles inside □ will be the lines dual to the polar point pairs inside □ and vice versa.

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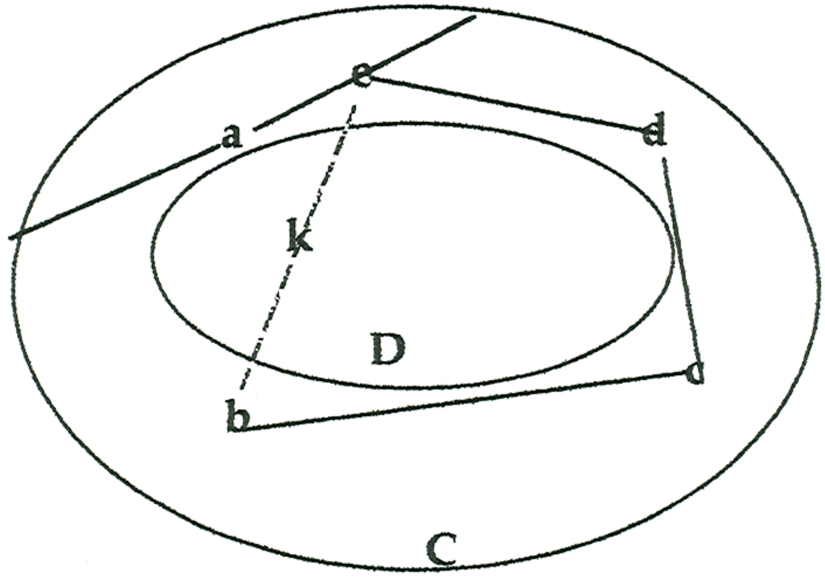
(d) Elliptic/Dual Hyperbolic Geometry

Project the region onto E^2 by the central map . The resultant planar region will be Euclidean space with a hole at the origin. Adjoining the line at infinity, (the projection of the equator) , and treating as line elements only those complete lines in E^2 which are not truncated by the hole at the center, one creates a model for elliptic/dual-hyperbolic geometry: no parallels, many seperals: any two points collinear to a line passing through the hole will be seperal. Topologically is a torus.

(e) Self-Dual Hyperbolic Geometry

Self -dual hyperbolic geometry can be modeled through simple modifications of preceding models : Let be, as before, stand for the interior of = (C,C*) .

Draw another convex loop, D , (with corresponding polar loop D′) inside and containing the North Pole . The space between C and D is an annulus, A . It is naturally self-dual: two lines (defined as segments of great circles truncated by both C and D) intersect in at most one point. Pairs of points are collinear to at most one line. Every point p in A supports a "cone" of lines, a truncation of the pencil of (truncated!) lines holding p. One must, of course , remove all lines , (such as k = (be) in the figure below) , from the line bundle on S passing through the region enclosed by D .



Self-dual Hyperbolic Geometry

Figure 8

(f) Pseudo Flat and Dual Pseudo Flat Geometries

The two remaining geometries, VI and VIII are familiar as representation spaces for theoretical physics. The appropriate projective constructions on S (or PPG) not so intuitive as the previous ones. VIII is also called ***pseudo-Euclidean geometry*** **, an expression which does not refer to its parallel/seperal structure but to the indefiniteness of its Riemannian metric. In our terminology, pseudo-Euclidean geometry is flat/dual-hyperbolic.**

On S:

Let g be some arbitrarily chosen segment of the equator H. less than □ in length so as not to intersect its polar equivalent g′. (This is not really a length restriction. It merely states that g is a piece and not the whole, of the equator , which is a proper projective concept.)

Remove from the line bundle all lines passing through the segments g and g′ **. Then remove H from the line bundle, and the points on H from the point space .**

This is pseudo-Euclidean geometry . When transferred to E^2 , one obtains a Minkowski space. There is a "prohibited cone" at each point p , derived from the lines passing between the poles and the segment g ; and a "permissible cone" , (with central angle □= □- g) , of the remaining lines through p . Following standard practice, the axis of the permissible cone is aligned with the vertical direction of time.

There is a slight difference in the models for Special Relativity depending on whether

(i) The segment g is open ended; or

(ii) g is closed

If g is a closed set, then the "light path " is in the prohibited cone, and cannot be attained on the permissible cone. This is the relativistic universe of matter alone. If g is an open set , then it may be possible to travel at velocity c, but not beyond. This is the geometry of matter plus radiation. Special Relativity indeed, distinguishes 3 distinct regions: time-like, space-like, and that upon the light cone itself.

This model can be extended. In fact, ***one is free to remove several segments g_1 , g_2 ,... , g_n*** **. from the equatorial line , H, together with the truncated pencils of all lines passing through them. The resulting space is still flat/dual-hyperbolic, yet now contains** ***several limiting velocities*** **, with "windows" of permissibility interspersed between them. Thus, one might be able to travel with a velocity less than c_1 ; greater than c_2 but less than c_3 ; greater than c_{2n} but less than c_{2n+1} ; and so on.**

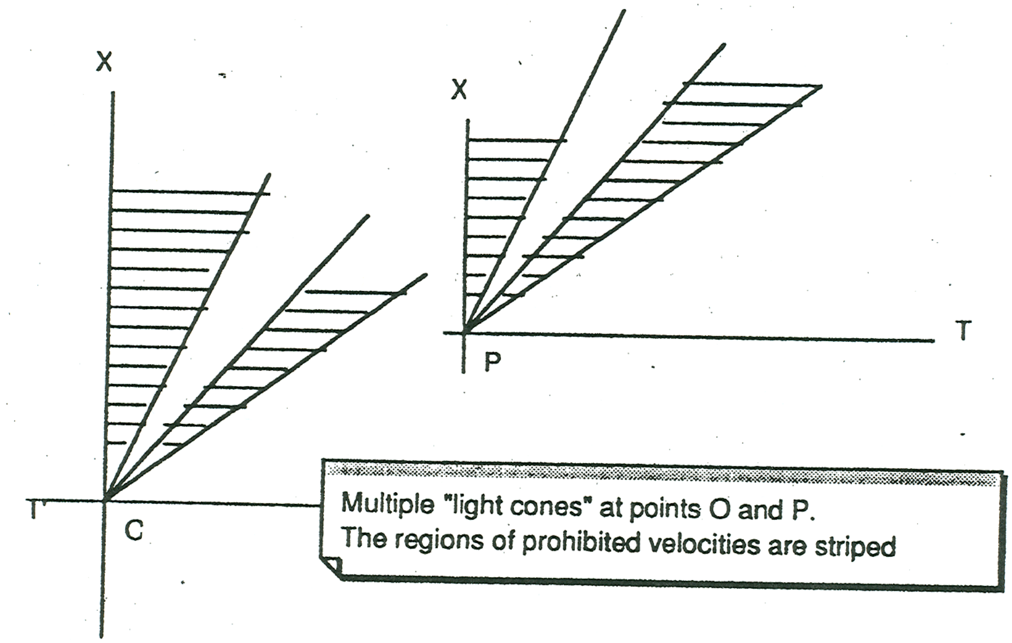


Figure 9

One is speaking projectively of course, not in terms of the Riemannian metric for proper time, which might become rather complicated. However, for a simple model involving two limiting velocities $u = c_{2n} < v = c_{2n+1}$, it is not difficult to construct a Riemannian metric that expresses the idea that velocities are constrained to be greater than u but less than v This metric takes the form:

$$\boxed{ds^2 = (dx - udt)(vdt - dx) = -dx^2 + dxdt(u+v) - uvdt^2}$$

One might conjecture from this that the 4 fundamental forces of nature could each possess a distinct velocity of propagation. The speed of the graviton might exceed that of the quantum! Perhaps it is possible to travel faster than gravity, but not at any speed between light and gravity, and so forth . Since this idea is consistently represented by a projective construct, it should not be dismissed without investigation.

(g) Dual-Minkowski Space

Our next model is that of dual - pseudo-Euclidean geometry. The North/South polar pair, □ , is dual to the equator, H. Dual to the segment (g, g′) on H is the pair of sectors , K=(J,J*) , between great circles meeting in □ and intersecting the equator with a separation g .

Therefore:

(i) Remove all the points in J and J′ from the point bundle . Remove all (north-south) longitudes composed of these points from the line bundle . The "lines" of this geometry are the truncations of all remaining great circles on S.

Transferring this construction to E^2 via the central map , a geometry is created inside a pair of wedges □ = (W, W′) .

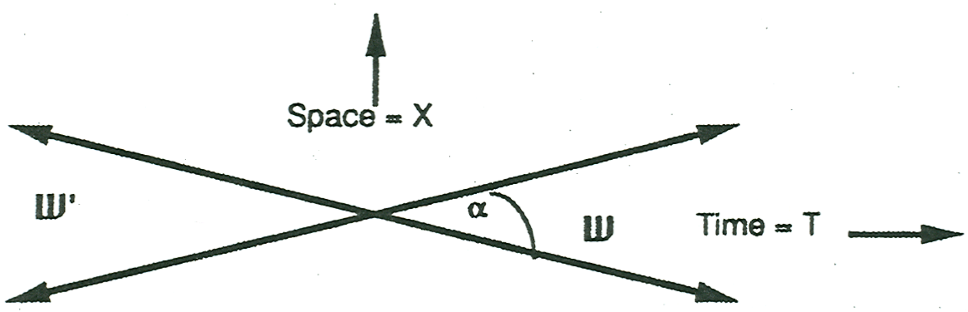


Figure 10

All of the lines inside □ that pass through the vertex have been removed: this engenders a 'unique seperals' property. The set of 'lines' inside the wedge pair, (the two parts of which are connected through the line at infinity) , consists of all segments inside □ (possibly connected through infinity) , truncated by at its defining walls.

Locate □ in the space-time plane so that it is bisected by the time axis, t . The space axis, x , then bisects the complementary region $□^C$. Thus , □ is "time-like", whereas $□^C$ is the 'proscribed' space-like region.

(h) Dual Relativity

Interpreting relativistic Minkowski Space, M^{1+1} , as a projective construct suggests the physical properties of corresponding entities in its dual. Let us review the features of Minkowski space:

(1) All points in the plane E^2 are retained.

(2) A sub-pencil of lines at each point is removed from the line-bundle .

(3) The remaining sub-pencils at each point subtend identical angles and have parallel boundaries.

(4) The line at infinity is also excised , allowing for a unique parallels geometry with corresponding (pseudo) Euclidean metric.

And here are the corresponding features of Dual Minkowski space, $M^{*(1+1)}$ in the wedge □= (W,W′) .

(1*) Retain every line in the line bundle of E^2 .

(2*) However only those points of a given line intersecting the interior of the wedge are retained.

(3*) Each line is then reduced to a segment, finite or infinite. One can place a metric on □ giving all such segments an identical *proper* length.

(4*) The vertex of the wedge is removed. However, since the line at infinity has been retained, the wedge is a single connected region. Lines of infinite extent, (that intersect the two edges of the wedge on opposite sides of the boundary), also connect at infinity.

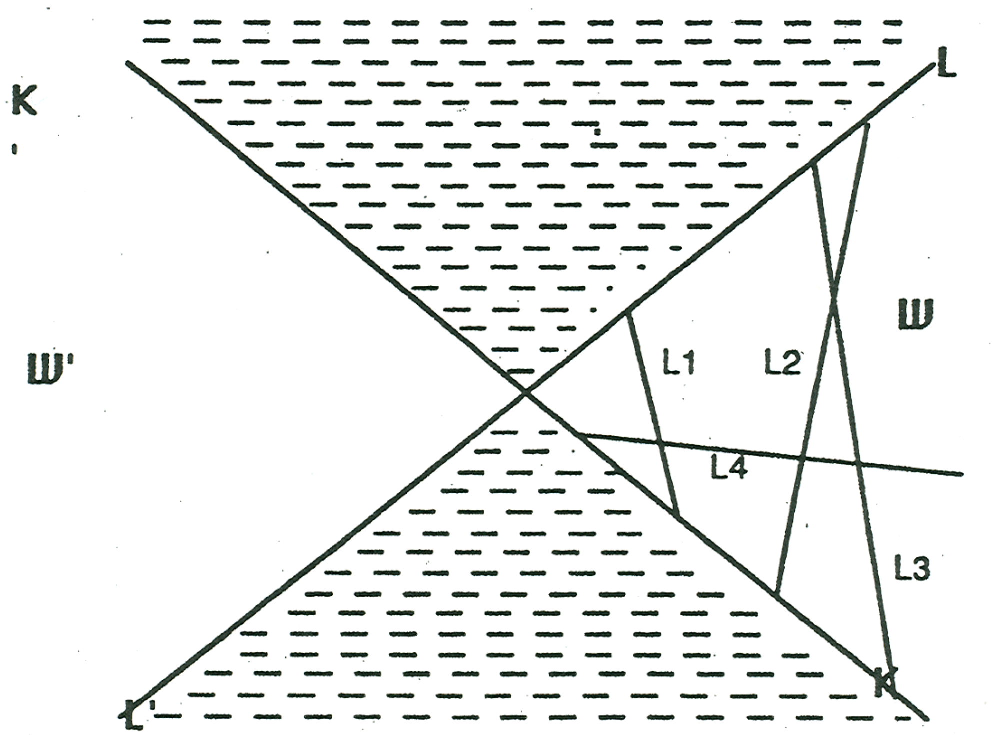


Figure 11

This is the model for *dual Special Relativity* . A possible physical interpretation of its geometric properties is :

(i) Although the vertex of the wedge is removed from space-time, since the line at infinity has been restored the wedge is a simply connected convex region.

(ii) One might interpret the vertex as the moment of the Great Explosion.

(iii) The trajectories of material bodies cannot pass through the origin. An observer located there can perceive nothing going away nor moving towards himself.

Equivalently, one might say that the straight lines emanating from the origin are purely temporal trajectories emerging from the Great Explosion. Observers on straight line trajectories passing through the origin cannot observe the passage of time, even as a person in a spatial location in Minkowski space has no absolute way of knowing if he is traveling or standing still. (The dual Relativity Principle .)

(iv) There do not exist space-time locations for which

$$\boxed{\begin{array}{l} x/t > g \\ g = \tan(\square) \\ \square = vertex \quad angle \end{array}}$$

(v) Dual to spatio-temporal location is the notion of a spatio-temporal trajectory. Trajectories inside the wedge can be interpreted as *photon paths terminating at the walls of the wedge* .

(vi) The points at which they terminate can be called *staticons* . Staticons move along the boundary at a velocity v determined by the Hubble expansion acceleration, $\square = v/T$.

(vii) As long as $v < c$, the speed of light , quanta will overtake the Hubble expansion and must therefore merge into staticons . Under the hypothesis that the Hubble velocity can never exceed light, all radiation becomes absorbed at the boundary by a staticon, (which may be equivalent to

the Hawking/Penrose interpretation of a Black Hole as a boundary point of cosmic space-time.) All radiation therefore begin and end in a staticon.

(viii) One may also state a dual Light Principle for dual-Minkowski space: *The distance from any point in the interior to any staticon is g.* **g is the proper distance which a quantum must travel before it reaches the boundary of the Hubble expansion .**

In the figure below Space-Time is the interior of the pair of wedges. The forward direction of time is off to the right, the forward direction of space towards the top of the page. Trajectories L_1 and L_2 have been drawn connecting the bounding walls. They have length "g" . The line connecting U with V has length less than g.

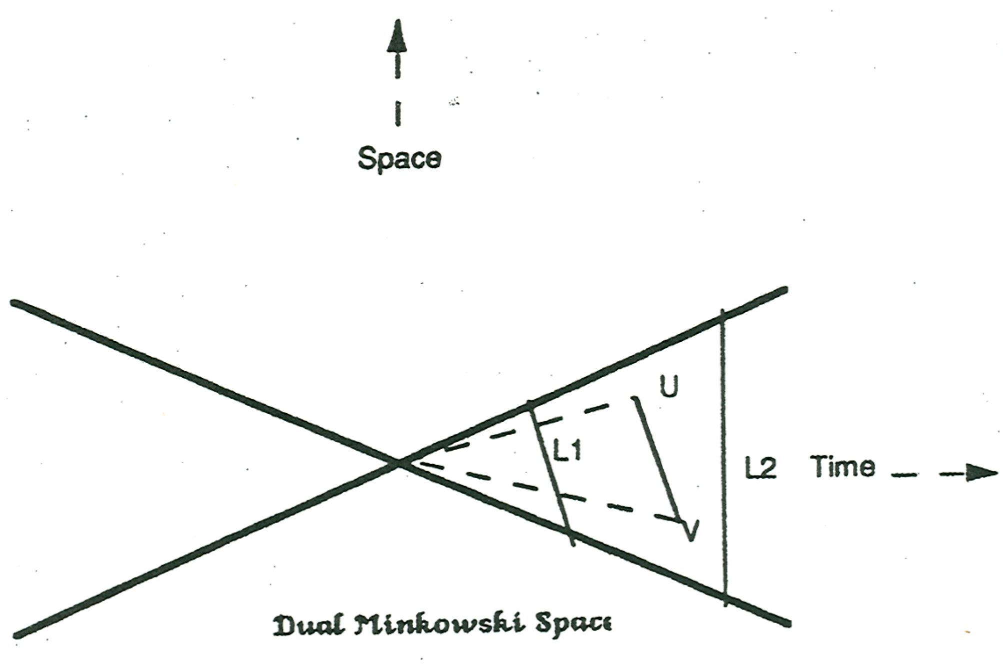


Figure 12

The similarities between this geometry and that of the Hubble expansion field are evident. Notice also how it contains a combination of both Big Bang and Steady-State evolution: some kind of "dual matter" is "born" at a staticon and "vanishes" at another, depending only on its velocity; this is the

Steady-State model. However, as the universe expands, new particles require longer periods of time to disintegrate.

Special Relativity, plus the dualization of Minkowski space, may turn out to be enough to derive and describe the Hubble expansion field. Further support to this hypothesis comes from this via a parametrization which "extends" the wedge over the entire plane. Finite line segments are mapped onto by infinite exponential spirals radiating outwards from a common origin.Their equations, in polar coordinates ρ, θ, are :

$$\boxed{\rho = e^{H(\theta+\mu)} \quad 0 \le \mu \le 2\pi}$$

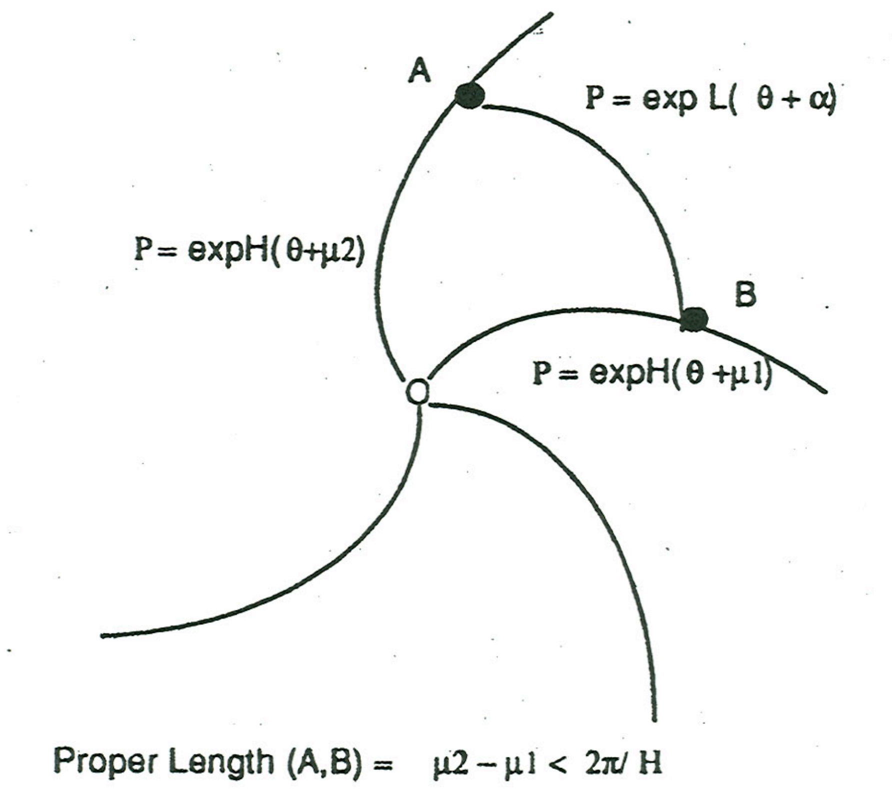


Figure 13

The points A and B lie on different world-lines emerging from the Great Explosion. The " proper length" between them is the rotation angle (or tangent thereof), between their two world-lines. The radius vector, ρ, may be identified with 'time', while the arc length of the unique exponential spiral

$\boxed{P = e^{L(\square+\square)}}$ **between them may be identified with 'distance', (appropriately normalized). Observe that no such spiral can be drawn when A and B are equidistant from the origin.**

The physical interpretation of this is the following: two events occurring at an identical 'world time' from the Great Explosion, cannot be connected by a space-time geodesic and are causally independent.

6. The Projective Postulate

Dualization of Minkowski space produces a dual-geometry. Physical entities modeled in the former representation space go over into entities in latter , which ought to turn up as observables in nature. The connection between Special Relativity and the Hubble expansion is suggestive, and appears to derive from a physical principle which we have named the Projective Postulate. Note its similarities to both the complementarity of Niels Bohr and Dirac's conception of anti-matter and bra-ket formalism. [24 -33] :

> **Projective Postulate**
>
> **" If the description of a natural observation is a projective construct, then its dual also exists in nature. "**

To say that an observable M is "projective" implies that, in the very act of its formulation, there is a *back-reconstruction* of a projective space P in which it is embedded . If its dual did not automatically exist by virtue of a co-dependent relationship, then M would not be truly projective. "Projective" concepts, by their very nature, arise only in dual pairs, in the same way that 'foreground/ background' , 'inside/ outside', 'more/ less' , etc. , cannot be defined independently.

Examples of Projective Constructs and their Duals.

The list of projective constructs in modern physics, with their projective duals, includes :

(i)

Construct: **The path of the quantum, considered as particle.**

Dual Construct : **The pencil of light rays about any unconstrained source of radiation, a wave phenomenon.**

(ii)

Construct : **The limiting velocity of the speed of light**

Dual Construct : **The Hubble expansion field.**

(iii) :

Construct : **The light cone as point space (particles)**

Dual Construct : **The light cone as line space (waves)**

(iv.):

Construct/Dual Construct : **Position Space/ Momentum Space dualism in the Operator/Hilbert Space treatment of quantum theory**

(v.):

Self- Dual Construct : **Uncertainty as quantity in Quantum Theory**

(See section 10)

Commentary :

(i) The metric geometry of Minkowski space allows one to treat the quantum world-line as a point: the proper time distance of any two events on its trajectory is 0. This is suggestive of the way in which the projective plane is constructed in 3-space, through the identification of each line through the origin of E^3 with a point on the surface of the unit sphere. A light cone is simply a plane of these "points" in 4-space intersecting the spherical surface

in a great hyper-circle, which may be treated as a wave packet emanating outwards in all directions from a light- source .

(ii) See section 5h.

(iii) A horizontal cross-section of the cone is a radiative light-pencil in 3-space. The ellipsoids arising from intersections of the light cone with horizontal hyper-planes function as 'hyper-lines' in a 3-dimensional projective geometry. This point of view is developed in Penrose's Twistor Geometry.

(iv) It is self-evident that the "phase space" of Quantum Theory is quite different from the "phase space" of Statistical Mechanics or Hamiltonian Dynamics. In the operator calculus of Quantum Theory one is free to describe phenomena in a "momentum formalism " or a "position formalism ", these descriptions being mediated by the Fourier Transform. In effect, the operator calculus of Quantum Theory acts over a projective space, with its co-dependent point and line bundles .

(v) The projective character of quantum uncertainty is discussed in the next section.

(8) Physical Applications

This completes the construction and classification of all Euclidean geometries derived from the negations , duals and dual-negations of the parallel postulate. Many of them already fulfill roles as representation spaces for modeling the phenomena of modern physical theory:

(1) *Elliptic/Dual-Elliptic :* The basic representation space for geology and astronomy. Spherical trigonometry

(2) ***Flat/Dual-Elliptic :*** **The stationary atemporal reference frame in the absence of gravitation. Euclidean geometry**

(3) ***Hyperbolic/Dual-Elliptic:*** **In Einstein's famous example of the "warping" of Euclidean geometry on the plane of a uniformly rotating disk, the metric geometry is hyperbolic. Gravitational fields create local hyperbolic geometries. Hyperbolic geometry also occurs frequently in Optics.**

(5) ***Flat/ Dual-Flat :*** **The self-dual phase space of Quantum Theory**

(6) , (7) . No present applications, although a suggested application for (6) is given in this paper.

(8) ***Flat/Dual Hyperbolic :*** **Special Relativity**

(9) ***Hyperbolic/ Dual Hyperbolic :*** **We will relate this to the geometry of quantum uncertainty.**

It would come as no surprise the author if it turns out that all of the non-Euclidean geometries are required for the modeling of the physical universe. The evidence seems to show that the universe is projective rather than metric in its fundamental geometric structure.

(9) Quantum Theory and Projective Geometry

In classical physics matter is a scalar magnitude, functioning more or less as a multiplicative parameter. In the statement of Newton's law of universal gravitation, the masses of attracting objects enter in the form of a simple product, linear in each . In the expressions for energy and momentum, mass is only a constant of proportionality.

Treating matter as a scalar means that , unlike space, (and in this century, time) , matter has no intrinsic ***geometry*** **. This despite the fact that , as was noted by Aristotle , material objects, (unlike temperatures, pressures and various other magnitudes) , always present themselves to our senses in the form of 3-dimensional** ***shapes*** **. However, the apparent solidity of matter, challenged already by Democritus, evaporates in elementary particle theory**

into a collection of force fields and binding connections between mathematical points.

Not being "geometrical", classical matter cannot be treated as a projective construct: substance has no dual correlative. Yet characteristics dual to those of matter were discovered in the behavior of radiation. These became incorporated in the idea of a field, described mathematically by James Maxwell. Fields, (propagated across regions of the universe) , are duals of particles, (best seen as compact systems in isolation .) Before the development of quantum theory , no connections were believed to pertain between the "radiating" field and the "punctual" particle.

Quantum Theory has connected space , time , matter and energy within a single self-dual flat geometry . Even as the localization ambiguities of light become incorporated in projective constructs in Special Relativity , so the localization ambiguities of matter are expressed through projective constructs in Quantum Theory. At the same time, the hyperbolic or Riemannian geometries appropriate to Special and General Relativity, and the dual position/momentum picture of Quantum Theory, are not compatible .

Observe that General Relativity does ascribe a kind of geometric structure to matter, via Mach's Principle for Inertia, and the Curvature Tensor interpretation of gravitation. In this instance "mass" functions as a kind of connection in the fiber above space-time as base. There is clearly a world of difference between the Space/Time/Matter geometry of Quantum Theory, and the Space/Time/Matter geometry of General Relativity.

It comes as no surprise that it turns out to be so difficult to combine these theories, as is attempted for example in Quantum Gravity. Not only must there be a way of casting all the laws of nature into a co-variant form, but every Observable implies the existence of a dual Observable. In line with the Projective Postulate, it would appear, in particular , that any successful resolution of the conflict between Relativity and Quantum Theory requires the existence of dual-relativistic Observables in a dual-Minkowski space.

(10) The Self-Dual Hyperbolic Geometry of Quantum Uncertainty

Let an object O, of mass m , be under observation, (such as an electron, whose mass is known in advance.) We wish to embed the measurement process on O in a 2-dimensional *uncertainty space* , J . The two dimensions of J are "time", t and "moment" y = mx , where x is position, (1-dimensional for convenience) . A point in J will plot our *knowledge* of the state of O. We will assume, without loss of generality , that O's state has been measured and that, within the latitude afforded by the Uncertainty Principle, has been shown to be "at rest" and " at the origin" of the frame of the investigator.

One way of expressing this is by stating: *We are certain that O lies something within a rectangular box , B , in J , with vertices (0, 0) ; (0,y) ; (t,0) ; and (t , y)* :

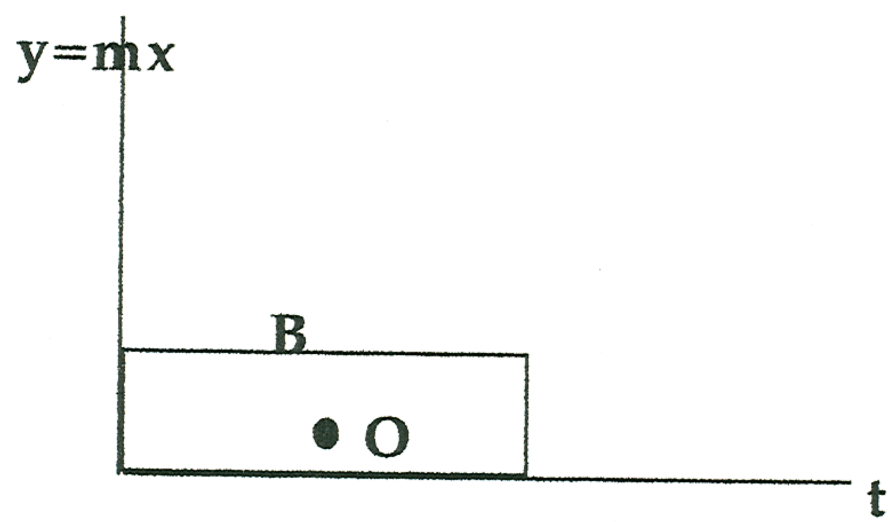


Figure 14

The actual values of t and y are given by the assumption that the trade-off in position and momentum errors must be greater than or equal to h/4 . If x (= y/m) is the error in position, then y′ = mdx/dt is the error in momentum.

Multiplying these gives

$$\boxed{x(\frac{mdx}{dt}) = \frac{y}{m}\frac{dy}{dt} \approx h/4\pi}$$

Replace the approximation by an inequality, and solve the corresponding differential equation for the *uncertainty parabola P* :

$$\boxed{y^2 = \frac{hmt}{2\pi}}$$

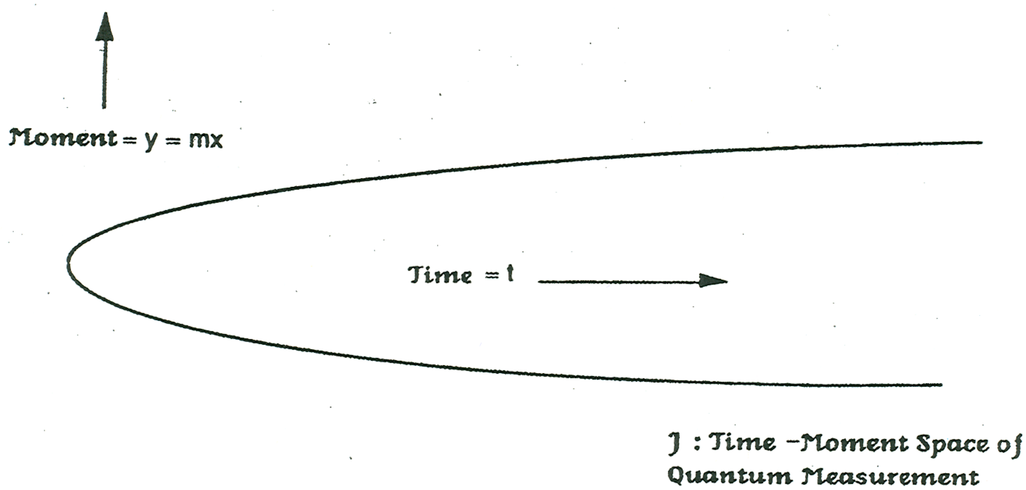


Figure 15

P can be interpreted in various ways :

(1) A statement like " O is at rest at the origin" , means that it lies *inside* the box B , yet *outside* the region bounded by P.

(2) The *uncertainty* , a *physical magnitude* associated with measurement with the dimensions of *action* , lies somewhere outside the region bounded by P .

Several analogies with the "light cone" are recognizable in this construction: the measurement of a massive object engenders an uncertainty parabola, P , emanating from the location in space-time from which the measurement was taken.

The parabolic paths with their vertex at (t,y) that lie outside P are paths about which some knowledge is possible. The parabolic paths inside P are not predicable, and no knowledge about them is obtainable.

Although these parabolas function, to an extent , like the lines of Minkowski space, one can have many parabolas which are " parallel" or non-intersecting, to a given parabola. Time-moment space, J, is therefore hyperbolic/dual-hyperbolic.

In the next diagram, the light-cone, y = (±) mct , has been superimposed on the uncertainty parabola:

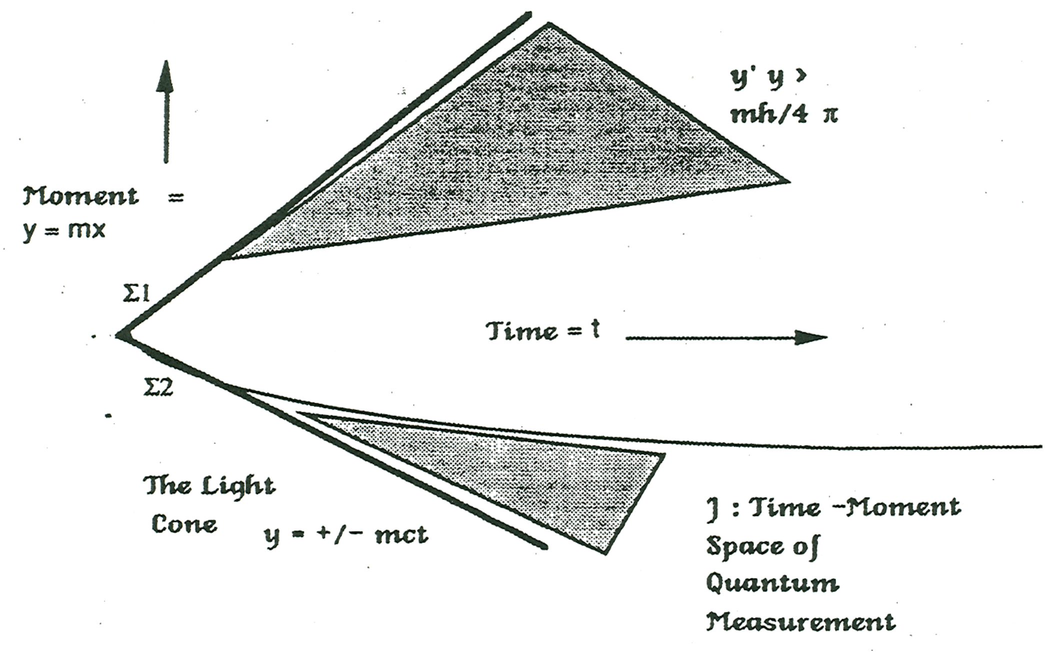


Figure 16

The regions of *possible knowledge* are in the grey areas between the cone and the paraboloid . Events inside the whitened region of the parabola are possible but unknowable. Events outside both the parabola and the light cone are knowable but impossible. Events in regions □1 and □2 , (*inside* the parabola yet *outside* the light cone, unfortunately completely covered by the dark lines of the drawing) *are both unknowable and impossible* .

The hyperbolic/dual-hyperbolic geometry J represents the extent of our knowledge of the state of O . Since all knowledge of the physical

universe derives from measurement it is reasonable to incorporate the geometry of J O itself. Mass, in other words , is more than scalar : its structure includes a hyperbolic/dual hyperbolic geometry. In section 5(d) this was shown to be the geometry on a torus.

Perhaps we *do* all live in a Yellow Submarine!

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