Special Lie Groups of Physics
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Introduction

The *group* is the most basic entity studied in the field of Abstract Algebra. Historically, the group notion arose from the investigation of symmetry transformations in Plane and Solid Geometry. It was then discovered by the brilliant young mathematician Evariste Galois, that the symmetric groups, or groups of permutations , could be applied to the solution of outstanding problems in the theory of algebraic equations. The so-called *Galois Group* has become the most important single tool in Number Theory.

The discovery of matrices by Cayley led to the theory of matrix representations of groups; this has many applications in modern physics. Sophus Lie discovered the great usefulness of continuous groups, now known as Lie Groups, and their associated Lie Algebras, to the solution of Differential Equations.

Emmy Noether, the most famous woman mathematician, demonstrated the close connection, indeed the equivalence, of groups of symmetries and the conservation laws of physics.

Dirac's equation for the electron, his theory of magnetic monopoles, the Standard Model of the electroweak force, and grand unified theories, all come directly out of the insights provided by Emmy Noether.

The important concept of the fundamental group of a topological surface was invented by Henri Poincaré, for which reason it is also known as the Poincaré group. This notion allows one to understand topological shapes in spaces of many dimensions. To give one example of a major application to modern physics, the Temperley-Lieb Algebra, encountered in the Ising Models of Statistical Mechanics, is derived from the Braid Group, a form of the Poincaré group that is central to Knot Theory.

Rotation Groups

We assume a basic background in group theory . Only certain Lie groups will be treated in this seminar. Generally speaking, these can be interpreted as generalizations of rotations in spaces of 2, 3 or 4 dimensions, parametrized by real or complex coordinates.

Although we will be speaking mostly about orthogonal groups, certain common Lie groups will also be defined now.

All Lie Groups of interest are groups of n- matrices with real or complex entries. There do exist Lie groups which cannot be represented as groups of matrices, but, at least for the purposes of physics, they can be dismissed as pathologies.

A matrix is a transformation on some n-dimension space. A Lie Group is a collection, M_k , of matrices acting on an n-manifold E_n , such that M_k is itself a continous k-dimensional manifold in its entries. This is best explained by examples. Let R^2 signify the ordinary Cartesian plane in 2-dimensions x, y.

A *non-singular* linear transformation A of R^2 sends each vector of the form v = (x,y) , into a vector w = $(a_1x + b_1y, a_2x + b_2y)$, where the coefficients are linearly independent. We can write A in the form

$$\boxed{A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}; Av = w; \det A = a_1b_2 - a_2b_1 \neq 0}$$

Since there are no restrictions on the entries in A, one can treat A as a continuous function of 4 variables. The collection of all matrices satisfying the above relationship can be parametrized as a subspace of R^4 subject only to the condition that detA not be equal to 0. This turns out to be all of 4-space, minus the hypersurface defined by the equation $\boxed{a_1b_2 - a_2b_1 = 0}$

The name given to this collection of matrices is the *General Linear Group* over the real numbers, of order 2 , or GL(R, 2) . For an n-dimensional space of real coordinates, one can similarly define GL(R, n) . Observe that this is always a group. The product of 2 matrices of GL(R,n) is also a member of GL(R,n) ; each element has an inverse; the identity is the matrix with 1's on the diagonal and 0's everywhere else, and since matrix multiplication is automatically associative, it obeys the associative law. Multiplication in GL(R,n) is not commutative, hence we say that it is a "non-Abelian" group.

If rather than detA 0 , one stipulates stronger condition detA = 1 , the result is another Lie group knowna as the *Special Linear Group* , or SL(R,2) (more generally SL(R,n)) . Note that, although the manifold of entry coefficients of SL (R,2) is a surface in R4 , it is actually a 3-dimensional manifold, because the condition $\boxed{a_1 b_2 - a_2 b_1 = 1}$ enables us to define any one of the entries in terms of the other 3. For example, $\boxed{a_1 = \dfrac{1 + a_2 b_1}{b_2}}$. As the determinant is this case is a quadratic form, the "shape" of this manifold in 4-space may be understood as a kind of "hyperquadric surface"

The coefficients need not be real. Transformations over complex spaces, such as the complex plane, will normally have complex numbers as entries. One then speaks of GL(C,2) , etc. Observe that GL(R,2) is a subgroup in GL(C,2)

The collection of 2-matrices A such that detA = 0, also forms a continuous 3-manifold in 4-space, but it is no longer a group. Elements of this collection do not have inverses.

Orthogonal and Unitary Matrices

Rotations and Reflections occupy a special place in the theory of Lie Groups. A rotation in n-space is a transformation that preserves the lengths of all vectors emanating from the origin. It can be called an isometry, or an orthogonal transformation.

Length in a real n-dimensional space is defined by the metric, the square root of a quadratic form in the dimensional variables. In the case of a general Hilbert Space the metric is derived from the norm. In the case of a general Riemannian Space, the metric is derived from something called the "connection"

The set of orthogonal matrices on 2-space is signified as O(2) .

If v =(x,y) is a generic vector in R^2 , the square of the length of this vector is defined as D =x^2 + y^2 . An orthogonal matrix is a linear transformation of v that preserves the value of D.

Recall these basic properties of matrices:

$$\boxed{\begin{array}{l} (AB)^T = B^T A^T \\ v = \begin{pmatrix} x \\ y \end{pmatrix}; v^T = (x, y) \\ v^T v = (x, y)\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 \\ (Av)^T = v^T A^T \end{array}}$$

Using these formulae one readily derives the defining condition for orthogonal matrices. Suppose O is an orthogonal matrix and $\boxed{Ov = w}$ **. Then** $\boxed{(Ov)^T = w^T = v^T O^T}$ **. Multiplying these together one sees that** $\boxed{w^T w = v^T O^T O v}$**. Therefore, if** $\boxed{O^T O = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$**, the identity matrix, then** $\boxed{w^T w = v^T v = x^2 + y^2 = D}$

This sufficient condition can also be shown to be a necessary condition. ***Therefore , O is an orthogonal matrix if and only if***

$$\boxed{O^T O = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

From our experience of working with rotations in the plane, we all know that they depend on a single variable, the included angle between a vector and the x-axis. Therefore, as a manifold, ***the Orthogonal Group in the plane is a 1-dimensional subspace of 4-dimensional space !***

The Orthogonal Group actually includes both rotations and reflections. A proper rotation is one for which detO = +1 . A rotation combined with a reflection has the property that detO = -1 . The collection of all proper rotations in the place is called the *Special Orthogonal Group* of order 2, written SO(2) .

Remembering our analytic geometry, we can notate a typical element of SO(2) as

$$\boxed{A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}$$

Labelling the coordinates of real 4-space as x,y,z,w the "equation" defining this curve in 4-space is x= w , y = -z , $x^2 + y^2 = 1$.

Looking as SO(3), SO(4), ... SO(n) , one discovers that the number of dimensions of the orthogonal group grows as a function of n. Observe that the embedding space of the coefficients of an n-by-n matrix will have n2 dimensions . Then it is not difficult to show that:

Theorem: The dimension of SO(n) as a manifold in n^2 dimensional space is m = n(n+1)/2 .

For our purposes we need only know that

SO(2) is 1-dimensional

SO(3) is 3 dimensional

SO(4) is 6-dimensional.

In particular, many of the sometimes confusing, yet always fascinating properties of spinors, quaternions and Pauli matrices that we will be discussing come from the fact that:

The number of free variables of the proper rotations in K= R^3 is equal to the dimension of K itself!

3-Space is the only Cartesian vector space with real coordinates for which this is true. It is this fact that makes it possible to have a vector product, which, as we know, is the most fundamental algebraic operation in Electromagnetism.

Complex Spaces

At some time in the 19th Century people began looking at spaces parametrized by complex numbers. The so-called Gauss-Argand-Wessel Diagram for the complex plane dates from 1797.

This is something of a private joke. In my review of Roger Penrose's book, "The Road to Reality", to be published by The Mathematical Intelligencer this summer, I make fun of Penrose's excessive pedantry in insisting on the priority of the otherwise unknown Caspar Wessel for the invention of he "complex plane". Not only did he not understand its larger implications, his paper wasn't even published until 1897 !

The *Complex Plane* C , or C^1 , is a way of assigning coordinates to ordinary plane geometry, such that the x coordinate is real, the y coordinate pure imaginary, and the entire location defined by a complex number z = x+iy. If z_1 , z_2 are complex numbers, one can also write them as

$$\boxed{\begin{array}{l} z_1 = \rho_1 e^{i\theta_1} ; z_2 = \rho_2 e^{i\theta_2} \\ z_3 = z_1 z_2 = \rho_1 e^{i\theta_1} \rho_2 e^{i\theta_2} = \rho_1 \rho_2 e^{i(\theta_1+\theta_2)} \end{array}}$$

The product of two complex numbers can be interpreted as a linear transformation in which the moduli are multiplied and the arguments added. In particular, if z_1 is located on the unit circle, then the product of z_2 by z_1 is a *clock-wise* rotation in the plane in the amount of the argument θ_1 of z_1. It is therefore natural to identify the complex numbers u = cosθ +isinθ of unit length with rotations. These can also be looked upon as the collection of *matrices of order 1* , (u) . This is known as SU(1) , the "Special Unitary Lie Group of Order 1". Observe that the elements u of SU(1) can be written in the form $\boxed{u = e^{i\theta} = \cos\theta + i\sin\theta}$

We have thus proven our first theorem:

Theorem:

SU(1) is isomorphic to SO(2) : $\boxed{SU(1) \cong SO(2)}$

Because multiplication by a complex number produces a clockwise rotation, the specific isomorphism connecting these two groups is $\boxed{e^{-i\theta} \leftrightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}$

It was primarily in the field of Projective Geometry that people began looking at spaces of nth order vectors, whose coordinates are complex numbers. The person most responsible for making this the standard way of studying projective spaces was the great mathematician and teacher Felix Klein. Klein deserves special mention, because it was the mathematicians who went to Europe to study with him who established the American tradition in mathematics.

The Unitary and Special Unitary Lie Algebras of Order n.

I will present their defining condition first, then explain the motivation behind it.

(1) *A Matrix U of order n is Unitary if*

$$\boxed{U * U^T = I_n}.$$

The asterisk signifies "U-conjugate", the matrix obtained by replacing all of its entries by their complex conjugates. This can written in several equivalent ways. Reviewing the basic properties of matrices : Let A,B be non-singular matrices of order n. Then :

$$\boxed{\begin{array}{l}(AB)^* = A^*B^* \\ (AB)^T = B^T A^T \\ (AB)^{-1} = B^{-1}A^{-1} \\ (A^T)^{-1} = (A^{-1})^T \\ \therefore ((AB)^T)^{-1} = (A^T)^{-1}(B^T)^{-1}\end{array}}$$

It follows that the defining condition for a unitary matrix can be variously written as

$$\boxed{\begin{array}{l}U^*U^T = U^TU^* = UU^{T*} = I_n \\ U^T = U^{-1*};(U^T)^{-1} = U^*\end{array}}$$

and so forth.

A *Special* unitary matrix V has the additional property that

$$\boxed{((V)) = \det V = +1}$$

Special unitary matrices differ from ordinary unitary matrices in the following respect: the determinant of an ordinary unitary matrix can be any complex number of modulus 1, such as i, -i , (-1+i 3)/ 2), etc..

SU(2)

Unitary matrices and the closely related unitary operators occur naturally in quantum theory. The time evolution of a Schrodinger wave function is given by

$$\boxed{\phi(t) = e^{-2\pi i/_h H}\phi(0) = U\phi(0)}$$

where U is a unitary operator. In the Heisenberg formulation these are in fact matrices.

Among all the Lie groups of unitary and special unitary matrices employed in physics, SU(2) has a unique place. We will show that :

(1) SU(2) is isomorphic to the quaternion group Q

(2) U(2) works as a group over spinors as O(3) does over 3-space

(3) The Pauli matrices $\boxed{\sigma_n, n = 1,2,3}$ are elements of SU(2) .

(4) SU(2) is a *double cover* for SO(3) . This means that for every matrix A in SO(3) , there are two matrices X, Y in SU(2) , and if
X_1 ,Y_1 correspond to A, X_2 ,Y_2 to B, then the products X_1X_2 , and Y_1Y_2 , correspond to C = AB .

(5) Each element E of SU(2) corresponds to a point p on the 3-dimensional surface of the unit sphere in 4-dimensional space. DetE=|p|=1. If E and F correspond to p and q , then EF =G corresponds to the product of p and q considered as quaternions.

Quaternions

It is claimed that Sir William Hamilton was standing on a bridge in Dublin staring over the Liffey river, when the idea of quaternions hit him in the head with the force of a shillalegh being thrown from an unknown source! Whatever the truth of the legend he spent many years in an attempt to recast all of Physics in the language of quaternions. This was not successful, and the subject was abandoned after his death .

With the discovery of quantum spin and other generalized rotation groups, quaternions were re-introduced into physics in the form of the Lie Group SU(2). This has been very successful.

A ***quaternion*** **q is a vector in (real) 4-space, written in the form**

$$\boxed{q = \alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3}$$

The letters i, j and k stand for square roots of -1, which relate under multiplication in the following manner:

$$\boxed{\begin{array}{l} i^2 = j^2 = k^2 = -1 \\ ij = -ji = k; jk = -kj = i; ki = -ik = j \end{array}}$$

Note the similarity with the vector cross product and with the curl. Large books have been written about the properties of quaternions : we present just one of them:

If $\boxed{Q = A^2 + B^2 + C^2 + D^2}$ **is a quadratic form in 4 variables, then it can be factored "over" the quaternions as**

$$\boxed{Q = (A + iB + jC + kD)(A - iB - jC - kD)}$$

We now examine their close relationship to Pauli matrices.

The Pauli matrices are defined by

$$\boxed{\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};}$$

We modify them very slightly, multiplying them by -i . Define

$$\gamma_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}; \gamma_2 = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \gamma_3 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

Then a simple calculation shows that

$$\boxed{\begin{array}{l}\gamma_1\gamma_1 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = \gamma_2\gamma_2 = \gamma_3\gamma_3 = -I_2\\ \gamma_1\gamma_2 = -\gamma_2\gamma_1 = \gamma_3\ldots.\end{array}}$$

These are exactly the defining equations for quaternions.

Unitary matrices as rotations

Let C signify an n-dimension space with complex

n

coordinates.

Consider the metric quadratic form defined by

$$\boxed{\begin{array}{l}x = (x_1, x_2, \ldots x_n)\\ D(x) = x_1^2 + x_2^2 + \ldots + x_n^2\end{array}}$$

Irrespective of whether the coordinates are real or complex, the preceding arguments can be used to show that the shape of this form is invariant under any linear transformation of the vector x by an orthogonal matrix O: $\boxed{D(O(x)) = D(x)}$

Furthermore : ***The metric quadratic form defined by***

$$\boxed{\begin{array}{l}x = (x_1, x_2, \ldots x_n)\\ D^*(x) = (x_1^2)^* + (x_2^2)^* + \ldots + (x_n^2)^*\end{array}}$$

is also preserved by orthogonal rotations. **This is because the application of "conjugation" does not reverse the order of the terms in a matrix product :** $\boxed{(OO^T)^* = O^*O^{T*} = I}$**. In this form the result is obvious. However, note that a vector is also a matrix. If we apply both conjugation and transposition to the product of a vector with a matrix, we get:** $\boxed{(Ov)^{T*} = (O^*v^*)^T = v^{T*}O^{T*}}$

Now consider the metric form :

$$\boxed{\begin{array}{l} x = (x_1, x_2, \dots x_n) \\ \Delta(x) = x_1 x_1{}^* + x_2 x_2{}^* + \dots + x_n x_n{}^* \end{array}}$$

Although the coordinates are complex numbers, the metric form is a real number! It is the simplest form of a Kähler metric, and a manifold with this metric is called a Kähler manifold. It is the natural n-dimensional extension of the modulus in C, the complex plane.

What class of matrices preserves the Kähler metric?

Let

$$\boxed{\begin{array}{l} v = (x_1, x_2, \dots x_n); v^T = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} \\ \Delta(x) = x_1 x_1{}^* + x_2 x_2{}^* + \dots + x_n x_n{}^* = v v^{T*} \end{array}}$$

We are looking for linear transformations U that preserve this form. What is needed is:

$$\boxed{\begin{array}{l} v^{T*} v = (Uv)^{T*}(Uv) \\ = v^{T*} U^{T*} U v \end{array}}$$

Looking at this equation one concludes:

$$\boxed{\therefore U^{T*} U = I_n}$$

We have shown that this condition is sufficient. It can also be shown to be necessary.

Therefore:

The Lie group of Unitary Matrices of order n preserve the metric on an n-dimensional Kähler manifold.

Likewise:

The Lie group of Special Unitary Matrices of order n preserve the metric and the gauge on a Kähler manifold.

Complex Manifolds, Orthogonal Matrices and Spinors.

For the present discussion we will restrict our attention to complex 3-space, C^3 . One isn't obliged to use a Kahler metric on a complex space. One can study the properties of the customary Euclidean metric D,

$$\boxed{\begin{array}{l} x = (x_1, x_2, \ldots x_n) \\ D(x) = x_1^2 + x_2^2 + \ldots + x_n^2 \end{array}}$$

where the x's are all complex numbers. In this case the metric preserving set of linear transformations will be the usual orthogonal group. Such manifolds are called "Euclidean"

It turns out that this Euclidean manifold relates to a Kähler manifold of lower dimension through the intermediary construction of something known as a Spinor manifold.

Observe that, as a Euclidean manifold, C^3 admits a new class of vectors, those of 0 length. These are known known as isotropic vectors.

An isotropic vector x **has the property that**

$$\boxed{\begin{array}{l} x = (x_1, x_2, x_3) \\ D(x) = x_1^2 + x_2^2 + x_3^2 = 0 \end{array}}$$

Apart from the trivial case of 0 vector, it is clear than at least one of the components must be a complex number. Let the subspace of isotropic vectors of C^3 be designated IS^3 Each isotropic vector element of IS^3 can be associated with a pair of vectors $\boxed{\xi = \pm(\xi_0, \xi_1)}$ in a Kähler 2-manifold known as a *spinor manifold* of order 2 : Sp(2) , The formal relationship of "isotropic vectors" to "spinors" is given by:

$$\boxed{\begin{aligned} x_1 &= \xi_0^2 - \xi_1^2 \\ x_2 &= i(\xi_0^2 + \xi_1^2) \\ x_3 &= -2\xi_0\xi_1 \end{aligned}}$$

The spinor components $\boxed{\xi = \pm(\xi_0, \xi_1)}$ are arbitrary real or complex numbers. A simple calculation shows that the Euclidean metric form on an isotropic vector is identically zero on Sp(2) under the above set of equations :

$$\boxed{\begin{aligned} &D = x_1^2 + x_2^2 + x_3^2 = (\xi_0^2 - \xi_1^2)^2 - (\xi_0^2 + \xi_1^2)^2 \\ &+4(\xi_0\xi_1)^2 \equiv 0 \end{aligned}}$$

What this shows is that every vector ξ of Sp(2) can be mapped onto a vector of IS^3 , and that every vector x of IS^3 corresponds to two vectors of Sp(2), $\pm\xi$.

However, something very interesting when one applies the *Kähler metric* **to IS^3 ! :**

$$\boxed{\begin{aligned} &\Delta = x_1x_1^* + x_2x_2^* + x_3x_3^* = (\xi_0^2 - \xi_1^2)((\xi_0^2)^* - (\xi_1^2)^*) \\ &+i(\xi_0^2 + \xi_1^2)(-i((\xi_0^2)^* + (\xi_1^2)^*) + 4(\xi_0\xi_1)((\xi_0\xi_1)^* \\ &= (\xi_0^2)(\xi_0^2)^* + (\xi_1^2)(\xi_1^2)^* - (\xi_1^2)(\xi_0^2)^* - (\xi_1^2)^*(\xi_0^2) \\ &+(\xi_0^2)(\xi_0^2)^* + (\xi_1^2)(\xi_1^2)^* + (\xi_1^2)(\xi_0^2)^* + (\xi_1^2)^*(\xi_0^2) \\ &+4\xi_0\xi_0{}^*\xi_1\xi_1{}^* \\ &= 2[(\xi_0^2)(\xi_0^2)^* + (\xi_1^2)(\xi_1^2)^*] + 4\xi_0\xi_0{}^*\xi_1\xi_1{}^* \\ &= 2(\xi_0\xi_0{}^* + \xi_1\xi_1{}^*)^2 \text{ (!!)} \end{aligned}}$$

The Kähler metric form on IS^3 is transformed into twice the square of the Kähler metric on Sp(2) . This means that the isometries of IS^3 are also the isometries of Sp(2), namely the unitary matrices , or SU(2).

We have arrived at the crucial step. When one goes from the Kähler Metric on IS(3) to the Euclidean Metric on IS^3 , one thereby

induces a natural mapping from the Special Unitary Group SU(2) onto the Euclidean Group of Special Orthogonal Matrices SO(3) . An orthogonal rotation in IS^3 translates into a unitary rotation in Sp(2).

This may be easier to understand when put into proper notation:

$$\boxed{\begin{array}{ccc} Sp(2) & ---> & IS^3 \\ \downarrow & & \downarrow \\ \Delta & ---> & D \\ \downarrow & & \downarrow \\ U(2) & ---> & O(3) \end{array}}$$

Let ξ be a spinor . It corresponds to a unique isotropic vector x. If x is translated into x' by means of an orthogonal transformation

x' = Ax, where A is a member of O(3) , then the spinors correspondings to x' , namely ± ξ ' , will be derived from by means of a unitary transformation U , ± ξ '= ±Uξ . A similar argument applies to the Special Orthogonal Group and the Special Unitary Group In the next section we will present the explicit relationship between the matrices of spatial rotations SO(3) and the matrices of SU(2).

SU(2), R4, Quaternions , and SO(3)

From the conditions which defines a member of the group SU(2), one can explicitely write down the form of each of its 4 entries. These conditions are:

$$\boxed{\begin{array}{l} (1)UU^{T*} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (2)\det U = +1 \end{array}}$$

Recall that, owing to the peculiar nature of multiplication of a matrix by a scalar, that det(U) =det(-U), when U is of even order!

The details of the calculation can be left as an exercise. One finds that a typical element of SU(2) can be written as:

$$\boxed{\begin{array}{l} V = \begin{pmatrix} \alpha_0 - i\alpha_3 & -\alpha_2 - i\alpha_1 \\ \alpha_2 - i\alpha_1 & \alpha_0 + i\alpha_3 \end{pmatrix} \\ \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \end{array}}$$

The a's are all real numbers. Let $\boxed{\sigma_1, \sigma_2, \sigma_3}$ signify the Pauli matrices. Then it is a simple exercise to show that V can be written in the form:

$$\boxed{\begin{array}{l} V = \alpha_0 I_2 - i(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3) \\ = \alpha_0 I_2 + Q \end{array}}$$

We recognize right away the form of a quaternion, as expressed in terms of the Pauli matrices. The imaginary part Q is what is called a pure quaternion, analogous to a pure imaginary in complex variables. Even as V is an element of SU(2), so Q is an element in su(2), the Lie Algebra associated with SU(2). Lie Algebras can be the subject of another talk. The function that carries elements V and -V onto an element R of SO(3) depends on the variables α_0 , ... This can be written out explicitely. If

$$\boxed{\begin{array}{l} V = \pm\begin{pmatrix} \alpha_0 - i\alpha_3 & -\alpha_2 - i\alpha_1 \\ \alpha_2 - i\alpha_1 & \alpha_0 + i\alpha_3 \end{pmatrix} \\ \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \end{array}}$$

then

$$\boxed{R = \pm\begin{pmatrix} \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 & 2(\alpha_1\alpha_2 - \alpha_0\alpha_3) & 2(\alpha_1\alpha_3 + \alpha_0\alpha_2) \\ 2(\alpha_1\alpha_2 + \alpha_0\alpha_3) & \alpha_0^2 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2 & 2(\alpha_2\alpha_3 - \alpha_0\alpha_1) \\ 2(\alpha_1\alpha_3 - \alpha_0\alpha_2) & 2(\alpha_2\alpha_3 + \alpha_0\alpha_1) & \alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2 \end{pmatrix}}$$

When $\alpha_0 = 0$, traceR = -1 and this scheme must be modified.

Details may be found in "Angular Momentum in Quantum Physics", L.C. Biedenham & J.D. Louck , "Encyclopedia of Mathematics, Addision Wesley 1981 . This book is in the Wesleyan Science Library.

Finally, because a quaternion of unit length can be expressed as a function of four variables connected by a Euclidean metric form, one can show that the topology of SO(3), considered as a continous topological manifold, is identical to that of the 3-Sphere in 4-dimensional real space, with polar opposite points identified, that is to say, the Projective Space of Order 3, or P^3 .

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