#1...

Dynamical Systems & Non-Linear Algebra

Roy Lisker

8 Liberty Street#306
Middletown,CT 06457
rlisker@yahoo.com
<http://www.fermentmagazine.org>

Part I

1. Introduction

This series of articles will look at connections between the general theory of algebraic structures, (commonly referred to as *Universal Algebra*) , and the modern viewpoint in the theories of *Dynamical Systems* , *Chaos* , *Fractals* and *Rational Maps* .

This preliminary investigation gives evidence in support the view that a rich mathematics - to which we give the name of *Non-Linear Algebra* - may be discovered in the interface of these two disciplines.

In the analysis of the dynamics of a function $y = \Phi (x)$ of a single variable, one isolates a sub-domain $\Lambda \varepsilon R$ - let's call these "interesting points" - to be tested for a certain specific properties. The interesting points are:

(1) The fixed points $\Pi = (p_1 , p_2, p_3 , ...)$, with the property $\Phi (p_j) = p_j$. This set may include $\pm \infty$.

(2) Pre-fixed or eventually fixed points $\Theta = \{ q_\alpha \}$. These have the property that for each of them there exists a k such that $\Phi^{(k)} (q_\alpha) = p_i \varepsilon \Pi$. The set Θ is closed under iteration : all numbers that iterate to elements of Θ are also in Θ. For example, if

$f (x) = x^2 + x - 1$, then the fixed points are -1 , 1 . Since f(0) = -1 , 0 is therefore a pre-fixed point, as are the solutions of f(x) = 0, $\beta_1 = (-1 + \sqrt{5})/2$, and $\beta_2 = (-1 - \sqrt{5})/2$; and the solutions

of f(x) = β_1 , f(x) = β_2, and so on. In general, the set of pre-fixed points for a given fixed point will be infinite, particularly if one admits the complex roots.

(3) The class of periodic points,

$$O = \{^{n_1} u_1, ^{n_2} u_2, ^{n_3} u_3, ... ^{n_j} u_j, ...\}$$

, where n_j equals the order of the periodicity of each point in O .

(4) The pre- , or eventually periodic points.

In addition to these there are:

(5) The chaotic points X ; those which, under iteration, neither converge to or equal any fixed or periodic point. In this paper, we will not be looking at this class.

Having determined that a point t is interesting , one then wants to know if it is attracting, repelling, neutral, the shape of its basin of attraction, the closure of its class, etc . As a general rule one is not so much interested in the behavior of a single function f , as one is in the properties of entire classes F of functions under the variation of one on more free parameters. The standard model for this procedure the logistic equation :

$\Phi(x, \lambda) = \lambda x(1-x)$, in which x is the variable, λ the free parameter.

The point of departure for this article is the question: What are the natural generalizations of these phenomena for polynomials of two variables? Take for example the function

$$F(x,y) = z = -y^2 + xy + 2y - x$$

The fixed points F (s,s) = s , are $s_1 = 1$, $s_2 = 2$. By substitution F(1,2) = 1; F(2,1) = 1; F(1,1) = 1 and F(2,2) = 2.

The values 1,2 therefore combine under F to give *a closed binary composition algebra* A , with multiplication table :

y/x	1	2
1	1	1
2	1	2

One recognizes this as the table in Boolean algebra for the conjunction "Or" , where "true" = 1, and "false" = 2.

Because the table is closed, every iteration g(x,y) of the function F, such as g_1 = F (x, F(x,y)) , g_2 = F (F(x,y), x),

g_3 = F (F(x,y), F(y,x)) , etc ., will give an output of either 1 or 2 for inputs 1, 2 in the variables x and y . Hence, each of the possible iterates g_k generates some binary composition algebra A , not necessarily conjunction, whose elements are specified by the values 1 and 2.

This example can be generalized to an entire class of surfaces determined by polynomials in 2-variables, of the form:

$$z = P(x,y) = \sum_{\substack{i,j=0,0 \\ i+j\leq n}}^{n,n} a_{ij} x^i y^j$$

On some of these, it which it may be possible to find various sets of values $Z_n^k = \{z_{n_1}, z_{n_2}, z_{n_3}, \ldots . z_{n_k})$

Under the operation of the function z = P (x,y) , Z_n forms a closed binary composition algebra, that is to say, that if z_r and z_s are in the set Z_n , then the value

z_{rs} = P (z_r , z_s) is also in the set Z_n . A = [Z_n , P] is called *k^{th} order binary composition algebra* , (there being no common term for it, other than "groupoid" which is not universally accepted). Its presence on the 3-dimensional surface created by P is called a representation of degree n of an k^{th} order composition algebra. Much of the discussion in this paper will turn around the relationship of order to degree.

POSITION STATEMENT: ***The natural generalization in two variables, of the k^{th} order periodic point for a function f of a single variable , is the k^{th} order binary composition algebra.***

2. Binary Composition Algebras: An Overview

Let E = { e_1 , e_2 ,......e_n } designate a set of n indeterminate symbols. We will say that E in combination with a binary relation , ° , is a binary composition algebra, or simply composition algebra, (or even just an "algebra" when there is no confusion) , if E is closed under ° : given u , v ε E , then the composition of u and v , w= f(u,v) = u ∘ v is also in E . There is no other condition. Binary Composition Algebras therefore include groups, semi-groups, monoids, and so on.

As a matter of convenience, one can write either w = f (u,v) or w = u ° v, or even w = uv when there is no confusion. However, since associativity isn't required one must be very careful in the placing of parentheses : (c (ab))d is very different from c ((ab)d) , and certainly from (ca) (bd).

Fundamental to the study of composition algebras are the multiplication tables. They are formed by three components: two vectors and a matrix:

T :

y/x	z_1	z_2	z_3	z_4
z_1	z_{11}	z_{12}	z_{13}	z_{14}
z_2	z_{21}	.	.	.
z_3	z_{31}	.	.	.
z_4	z_{41}	.	.	.

The *horizontal border* b_H is the vector

(z_1 z_2 z_3 . . z_m) containing some sub-set, or the entirety, of the elements of the algebra E in some order . (The z's range over

the elements of E) It represents the y-coordinate in z= f(x,y) . We allow for repeated elements. The vertical border b_V is the transpose vector:

$$bV = \begin{matrix} z_1 \\ z_2 \\ z_3 \\ . \\ . \\ z_l \end{matrix}$$

It contains some or all of the elements of E, with perhaps some repetitions, and need not be in the same order as the horizontal. When the horizontal and vertical borders contain all and only the elements of E, only once, and are in the same order, then we call T a *standard table* , or, simply, *the* table of E .

The body of a standard table, M , called it's *matrix* , is the collection of double-indexed terms $z_{ij} = z_i \circ z_j$, located in the row initiated by the term z_i in the vertical border and the column initiated by the term z_j in the horizontal border. can also be called the matrix of the algebra itself, as all other matrices can be derived by permutations of rows and columns of M .

Clearly it is possible for standard tables of two algebras to have the same matrix, yet be quite different depending on the contents on their borders. Consider these examples:

U:

y/x	a	b
a	a	b
b	a	b

V:

y/x	b	a
b	a	b
a	a	b

The matrix for each of these tables is the same, but both horizontal and vertical borders have been reversed. Notice that the first algebra has two fixed points, whereas the second algebra has no fixed points: they are not isomorphic. One can also easily construct two algebras with the *same* vertical and horizontal borders, with quite *different* matrices, which *are* isomorphic. Determining if two algebras are isomorphic is not always a simple matter. In a paper written in 1987, I give an algorithm for determining, from a standard table, if its' algebra is associative, that is, if it is a semi-group.

When b_V and b_H include all the elements of Z, then the structure of its composition algebra is completely given by the corresponding table . Symbolically, we can notate this as:

$$f: Z \otimes Z \to Z \equiv (Z, P, \circ) \equiv (b_V, b_H, M)$$

One becomes interested of course in obvious combinatorial questions like, the features common to algebras whose tables have identical matrices , (as in the above example) ; algorithms for identifying isomorphic algebras; or identifying semi-groups; the groups of symmetries over these tables, (essentially Galois groups) , and so forth. A useful feature of the tabular presentation is that it is a visual format that may be physically composed with itself to produce the tables of the successive iterates of the function which governs it. This example shows what I have in mind:

$$f(x,y) \rightarrow f(x, f(x,y))$$

y/x	z_1	z_2	z_3
z_1	z_{12}	z_{12}	z_{13}
z_2	z_{21}	z_{22}	z_{23}
z	z_{31}	z_{32}	z_{33}

$\rightarrow$

z_1	z_{11}	z_{12}	z_{13}
	z^1_{11}	z^1_{12}	z^1_{13}

+

z_2	z_{21}	z_{22}	z_{23}
	z^2_{21}	z^2_{22}	z^2_{23}

+

z_3	z_{31}	z_{32}	z_{33}
	z^3_{31}	z^3_{32}	z^3_{33}

$\downarrow$

y/x	z_1	z_2	z_3
z_1	z^1_{11}	z^1_{12}	z^1_{13}
z_2	z^2_{21}	z^2_{22}	z^2_{23}
z_3	z^3_{31}	z^3_{32}	z^3_{33}

The 3 rows of the matrix of an algebra A = [Z , f(x,y)] are composed successively with the elements of the vertical border to produce the matrix of [Z , f(x,f(x,y))]. The horizontal border is then brought down from the original table to make the standard table for the new algebra A' = [Z, f(x,f(x,y))] .

Iterating f in all possible ways one derives a set of related functions, which generate other algebras and tables. If f is a arbitrary function in two variables x, y closed over a given finite range E, then the collection of all iterates of f is a second order clone, which we call the *iterate collection* ,

Ψ_f. The elements of Ψ_f are f(x,x), f(x,y), f(y,y), f(y,x) , f(x,f(x,x)), f(y,f(x,x,)), ...

We will also be interested in the collection of functions Ψ_f^c , where c is an indeterminate constant (or collection of constants $c_1, c_2,...c_n$), and the iterate collection is enriched by the functions f(x,y) = c, f(c,x), f(x,c), f(c, f (x,y)), f(x, f (c,y)) ,..... This may be called the iterate collection with constant c .

The terms inside the brackets can be called monomial forms. The *free algebra of bivariate forms* , $M_{(x,y)}$, which is also a clone, is given by the collection of strings: (x) , (y) , (xy) , (x(xx)), ((xx)x), ((xx)(xx)) , (y(xx)), (xx)y , ((xy)(xy)), ((x(yx))y) .... M is related to Ψ_f through the procedure of adding a symbol "f" to the left of each left parenthesis, and by placing commas between successive entries of the variables , and between juxtaposed right and left parentheses :

)(becomes),(, etc.

Another useful construction is that of the *free algebra of monomial forms* , Ξ . This is constructed recursively by

(i) $\pi_0 = x ; \pi_1 = y$

(ii) . We extend the domain of the function f(x,y) which is over $E \otimes E$, to include E itself, on which it is the identity:

$f(e_j) = e_j$ for all $e_j \in E$

(ii) If $\pi_\alpha , \pi_\beta \in \Xi$, then $\pi_\gamma = (f(\pi_\alpha) , f(\pi_\beta)) \in \Xi$

In other words, each π is an ordered pairs of functions of Ψ_f , and each function in Ψ_f may be interpreted as the application of f, to some argument π(x,y). Once again the description can be enriched by the addition of one or more constants c_i .

The construction of Ξ gives us no information on the manner in which the monomial forms are counted . Although there is no unique method of enumeration , there is only one way which uses all of the natural numbers

only once, and which gives $\pi_\gamma = (f(\pi_\alpha), f(\pi_\beta))$ a higher index than either π_α or π_β : the Cantor function on pairs of integers. It will be described in a moment.

Suppose now that we are given some closed binary algebra A = (Z, f), Z = (z_1 , ...z_n) , in which n is finite: consider the collection of iterates of f over A . Assuming a method for counting the monomial forms, we can identify each function in Ψ_f by its index:

$$f_j \equiv f(\pi_j(x,y)) .$$

Each such function generates its own composition algebra over Z, closed since f(x,y) is closed, with its own table, T_{π_j} . Since the cardinality of Z is n, and since each of the n^2 entries in T can be freely chosen, it follows that there a total of n^{n^2} tables that can be formed from Z. The number of *distinct* tables generated by the iterates of f will be a sub-class of these, in general a much smaller number. It follows that the number of distinct elements of Ψ_f over a given finite composition algebra A , is finite. We will call this finite set of algebras the algebraic class generated by A , or the elaboration of Ψ_f over A , or simply, the class of A .

If the functions and monomial forms have already been enumerated, this construction sets up a natural congruence structure over the integers: Writing our algebra as $A^{(n)}$, to indicate the order of A , we write $f_j \equiv f_k \bmod(A^{(n)})$, or just $j \equiv k \bmod(A^{(n)})$ to signify that these two functions give identical tables over Z . The integers j and k are determined only by the method of enumeration , whereas the actual structure of the congruence will depend on both the number n and the original function f .

It is required, of course, that f_j and f_k generate exactly the same table, not merely isomorphic ones .

A Non-trivial Example

We will say that an algebra A $^{(n)}$ is "right insensitive" (or "left insensitive"), if for all u,v ε Z f(u,v) = f(u,u) ,

(respectively, = f(v,v)) , that is to say, f is essentially a function of the left (right) variable alone . Restricting our attention for the moment to second order algebras only, Z = Z_2 = (0, 1), we adopt a certain enumeration scheme for the elements of Ξ , and examine *the simplest of the right insensitive algebras of the second order* , namely, the one generated by the function

f(x,y) = x ε Z_2 .

This function is represented by the table

y / x	0	1
0	0	0
1	1	1

What the structure of the congruence generated, relative to the Cantor enumeration, over the integers? We will see that it is fairly complicated. The following observation gives us some idea of what it looks like:

Let π_α be some element in Π other than π_0 or π_1 . It will then be an ordered pair $\pi_\alpha = (f(\pi_\alpha^\lambda), f(\pi_\alpha^\rho))$

and we can write :

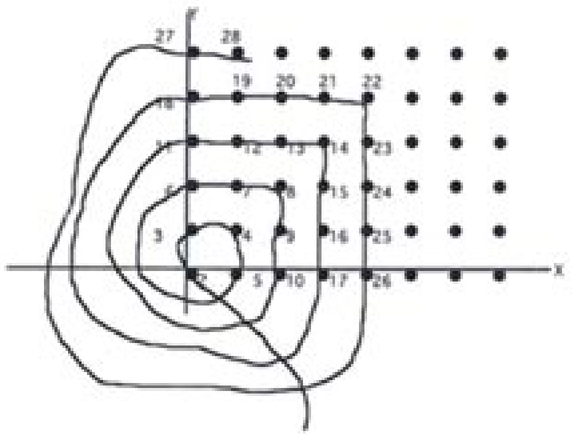
$f_\alpha \equiv f(\pi_\alpha) = f(f(\pi_\alpha^\lambda), f(\pi_\alpha^\rho))$ **. If f is right insensitive, the value of $\pi_\alpha{}^\rho$ will be irrelevant: the table for π_α will be the same as that for $\pi_\alpha{}^\lambda$. Proceeding in this way, *always eliminating the right most monomial form in the reduction of π_α* , we see , finally , that the value of f_α depends only on whether the left-most element of π_α is x or y. Our congruence will therefore be of the form**

$$k \equiv 0 \text{ or } k \equiv 1 (\text{mod } f(x,y) = x, x \varepsilon Z_2)$$

3. The Standard Enumeration of the Free Monomial Algebra Without Constants

#11...

The method of enumeration which we will use is based on the correspondence between the spiral and the square that was devised by Georg Cantor in his proof that the ordinal of the set of rational numbers is countable.



Cantor Enumeration of Free Monomials

The diagram shows the positive , upper-right quadrant of the real Cartesian plane. The points representing the ordered pairs (m,n) are marked. The numbers on the diagram enumerate the 0-dimensional complex formed by this collection of points by wrapping a spiral *clockwise* through them . *Note that the enumeration begins with 2.* We will see why this is so in a moment.

Define : $\pi_0 = x$, $\pi_1 = y$. The enumeration scheme substitutes back and forth between (0,1) and (x,y). Starting from (0,0) as point 2 , count and label all the points on the network up to the j^{th} point on the spiral. This number corresponds to an ordered pair (l , r). Since both l and r are clearly less than j, we can locate the l^{th} and the r^{th} points on the spiral. These correspond to ordered pairs (l_l , l_r) and (r_l , r_r) . Identifying the j^{th} point with the monomial form π_j , write:

$$\pi_j = (f(\pi_l), f(\pi_r))$$
$$= (f(f(\pi_{l_l}), f(\pi_{l_r})), f(f(\pi_{r_l}), f(\pi_{r_r})) = \ldots$$

How the enumeration works in practice will become clear from the calculation of the first 20 monomial forms. Setting up the correspondences in the form of a table , we have:

#13...

#14...

$(0) \leftrightarrow 0 \leftrightarrow x$

$(0,1) \leftrightarrow 3 \leftrightarrow (x \cdot y)$

$(0,2) \leftrightarrow 6 \leftrightarrow (x \cdot (x \cdot x))$

$(2,1) \leftrightarrow 9 \leftrightarrow ((x \cdot x) \cdot y)$

$(1,3) \leftrightarrow 12 \leftrightarrow (y \cdot (x \cdot y))$

$(3,2) \leftrightarrow 15 \leftrightarrow ((x \cdot y) \cdot (x \cdot x))$

$(1) \leftrightarrow 1 \leftrightarrow y$

$(1,1) \leftrightarrow 4 \leftrightarrow (y \cdot y)$

$(1,2) \leftrightarrow 7 \leftrightarrow (y \cdot (x \cdot x))$

$(2,0) \leftrightarrow 10 \leftrightarrow ((x \cdot x) \cdot x)$

$(2,3) \leftrightarrow 13 \leftrightarrow ((x \cdot x) \cdot (x \cdot y))$

$(3,1) \leftrightarrow 16 \leftrightarrow ((x \cdot y) \cdot y)$

$(0,0$

$(1,0$

$(2,2) \leftrightarrow$

$(0,3) \leftrightarrow$

$(3,3) \leftrightarrow$

$(3,0) \leftrightarrow$

Consider the spiral point #9. This corresponds to the ordered pair (2,1). Point # 2 is at the origin and corresponds to (0,0). Since 0 corresponds to "x" and 1 to "y", it follows that the 9th monomial form, or π_9 = (f (0,0), f(1)) = (f(x,x), y) which is the same as ((x.x).y) in composition notation. Then the 9th function in the iterate collection is f (π_9) = f(f(x,x),y). This description may appear a bit repetitive, but it helps to see the enumeration scheme as being applied to the monomial forms.The natural extensions to the iterate collection and the bivariate algebra are self-evident.

Any ordered pair of integers can thus be corresponded to a monomial form. For example:

(9,7) -> ((2,1), (1,2)) -> ((0,0), 1), (1,(0,0))-> ((x,x),y),(y,(x,x))

-> f(f (f (x,x),y)), f(y,f(x,x))) .

By counting along the Cantor spiral we discover the the index of the ordered pair (9,7) is 9^2 + 10 + 2 = 93. The above is therefore the form of f(π_{93}) .

Describing the algorithm in general terms: The index N corresponds to the N-1st point on the clockwise Cantor spiral. This is an ordered pair, (L,M). L corresponds to point L-1 on the spiral, which is an ordered pair, (A,B) , M to point M-1, which is an ordered pair (C,D). This process is continued until every index has been expanded into a nested expression involving only 0's and 1's. Replace 0 by x and 1 by y, placing the functional symbol f to the left of each left parenthesis, replacing each dot by a comma gives us the Nth monomial in Π.

Letting $\Im$ stand for index function, then

$\Im$(a,b) = n , where p= (a,b) is an ordered pair in the plane and n expresses the fact that p is the n-1st point along the spiral. The domain of $\Im$ consists of

(i) the two integers 0 and 1, and

(ii) the set of all integer pairs in the bordered upper right hand quadrant of the real plane. Its range is of course, Z_0^+, the natural numbers with 0 .

Through an examination of Figure 1 it is a simple matter to write down the algebraic equation for $\Im : \Im(a,b) = n = \begin{Bmatrix} a + b^2 + 2; a \le b \\ (a+1)^2 - b + 1; a \ge b \end{Bmatrix}$

Since this function is 1-1, it has an inverse: Consider first the case n = a + b^2 + 2 a ≤ b . Then

b^2 < n -2 < $(b+1)^2$. Therefore:

$$b = [\sqrt{n-2}], a = n - 2 - b^2$$
$$= n - 2 - [\sqrt{n-2}]^2$$

Since we are assuming a ≤ b , this works out to

$$n \le 2 + [\sqrt{n-2}] + [\sqrt{n-2}]^2 \equiv \phi(n).$$

Otherwise we have:

$$n > \phi(n), n = (a+1)^2 - b + 1,$$
$$a = [\sqrt{n-2}], b = (a+1)^2 - n + 1$$
$$= [\sqrt{n-2}]^2 + 2[\sqrt{n-2}] - n$$

Summarizing:

$$\Im^{-1}(n) = \begin{Bmatrix} (n - 2 - [\sqrt{n-2}]^2, [\sqrt{n-2}]), n \le \phi(n) \\ ([\sqrt{n-2}], ([\sqrt{n-2}]^2 + 2[\sqrt{n-2}] - n), n \ge \phi(n) \end{Bmatrix}$$

By way of illustration we elucidate the congruence relations within the free bivariate algebra, A(mod f(x,y) = x) over the domain Z_2 = (0,1) . In general

f_k = f(π_k (x,y)) = x

if the left-most variable in π_k is x ,or y if it is y. Inspecting the first few numbers in Figure I one has:

$$0 \equiv 0,2,3,6,8,9,10,11,12,13,14,15,16,17,,,$$

$$1 \equiv 1,4,5,7,22,23,24,25,26,........$$

In general f(π_k) = (f(π_l), f(π_r)) = f(π_l) .

From the formulae for the Cantor indexing function, it follows that :

$$l = \begin{cases} (k-2-[\sqrt{k-2}]^2), k \leq \phi(k) \\ ([\sqrt{k-2}]), k \geq \phi(k) \end{cases}$$

Successive iterations of l via these formulae produces a sequence l(k) , l(l(k)), l(l(l(k)) , a composition exponent is reached for which $l^{(q)}(k)$ = either 0 or 1. One can show that q is the smallest integer for which $1 < (k-2)^{1/(2^q)} < 2$ **. Solving for q we obtain** $q = \ln_2(\ln_2(k))$**. We can therefore represent the congruence in the form:**

$$k \equiv [l^{(1+\ln\ln_2 k)}(k)](\text{mod } f(x,y) = x, x\varepsilon Z_2)$$

Although this formula is very cumbersome, there are simpler relations which show us what is actually going on: By the nature of the left identity function f, $k \equiv \Im(k,h)$ **for all h.**

Let k ≤ h ; then

$$k \equiv k + h^2 + 2(\text{mod } f). Writing$$

$$h = k + m, gives$$

$$k \equiv k + (k+m)^2 + 2 = k^2 + k(2m+1) + m^2 + 2, \forall m \geq 0$$

Similarly for k>h ,we have

$$k \equiv (k+1)^2 - h + 1 = k^2 + 2k + 2 - h(\text{mod } f).$$

These formulae quickly generate the residue classes C(0) and C(1) by plugging in various values for k, h and m. Thus, starting with m=0 k= 0 we have

$$0 \equiv 2 \equiv 8 \equiv 74 \equiv ...$$

$$m = 1, k = 0 \rightarrow 0 \equiv 3 \equiv 21 \equiv ...$$

$$m = 0, k = 1 \rightarrow 1 \equiv 4 \equiv 22 \equiv ... etc.$$

4. Binary Composition Algebras of Order 2

For fixed standard borders, the table of a composition algebra of order 2 has four entries. Therefore the number of distinct tables is $2^4 = 16$. After quotienting by isomorphism one is left with only 7 distinct composition algebras of order 2. They are listed here, grouped by isomorphic forms.

We let x and y stand for variables , z_1 and z_2 for indeterminates, or unspecified real or complex numbers :

1. The constant algebras:

$O_{z_1}: f(x,y) = z_1$

y/x	z_1	z_2
z_1	z_1	z_1
z_2	z_1	z_1

$O_{z_2}: f(x,y) = z_2$

y/x	z_1	z_2
z_1	z_2	z_2
z_2	z_2	z_2

These algebras are obviously isomorphic.

2. The right (left) insensitive algebras with two fixed points:

#19...

$L_x: f(x, y) = x$

y/x	z_1	z_2
z_1	z_1	z_1
z_2	z_2	z_2

$Ly: f(x, y) = y$

y/x	z_1	z_2
z_1	z_1	z_2
z_2	z_1	z_2

3. The right (left) insensitive algebras without fixed points:

$k = z_1 + z_2$;

$B_x: f(x, y) = k - x$

y/x	z_1	z_2
z_1	z_2	z_2
z_2	z_1	z_1

$By: f(x, y) = k - y$

y/x	z_1	z_2
z_1	z_2	z_1
z_2	z_2	z_1

4. The group, G :

#20...

G_{z_1}:

y \ x	z_1	z_2
z_1	z_1	z_2
z_2	z_2	z_1

G_{z_2}:

y \ x	z_1	z_2
z_1	z_2	z_1
z_2	z_1	z_2

This is the only second-order group.

5. The Boolean Algebras:

$\wedge$("and"):

y \ x	z_1	z_2
z_1	z_1	z_1
z_2	z_1	z_2

$\vee$("or"):

y \ x	z_1	z_2
z_1	z_1	z_2
z_2	z_2	z_2

6. The 'Anti-Boolean Algebras':

X_{z_1}: ("¬(p ∧ q)")

y \ x	z_1	z_2
z_1	z_2	z_1
z_2	z_1	z_1

X_{z_2}: ("¬(p ∨ q)")

y \ x	z_1	z_2
z_1	z_2	z_2
z_2	z_2	z_1

7. The 4 Implication Algebras.

R_1:

y \ x	z_1	z_2
z_1	z_2	z_1
z_2	z_2	z_2

R_2:

y \ x	z_1	z_2
z_1	z_2	z_2
z_2	z_1	z_2

R_3:

y \ x	z_1	z_2
z_1	z_1	z_2
z_2	z_1	z_1

R_4:

y \ x	z_1	z_2
z_1	z_1	z_1
z_2	z_2	z_1

All of algebras are isomorphic, and represent, respectivally, the truth tables for p->q, q->p, ¬(p-> q), and ¬(q-> p).

5. Representations of Second Order Algebras by Linear Forms

All of the insensitive algebras can be represented by functions of a single variable:

$$O_{z_1}: f(x,y) = z_1$$
$$O_{z_2}: f(x,y) = z_2$$
$$L_x: f(x,y) = x$$
$$L_y: f(x,y) = y$$
$$B_x: f(x,y) = k - x; (k = z_1 + z_2)$$
$$By: f(x,y) = k - x$$

Linear representations are not possible for the other algebras:

THEOREM I: No linear form $l(x,y) = Ax + By + C$, with both A and $B \neq 0$, can represent any composition algebra of order ≥ 2.

The theorem only needs to be proven for second order algebras, the proof being similar for higher orders. Let K be any one of the six 2nd order algebras, and write its table in the form:

$$g(z_1, z_1) = z_{11}$$
$$g(z_1, z_2) = z_{12}$$
$$g(z_2, z_1) = z_{21}$$
$$g(z_2, z_2) = z_{22}$$

At least two of these expressions must be equal to each other. By the inherent symmetries we need only consider two cases:

(i) $z_{11} = z_{12}$

(ii) $z_{11} = z_{22}$ and $z_{12} = z_{21}$

In the first case we see that

$$g(z_1, z_1) - g(z_1, z_2) = A(z_1 - z_1) + B(z_1 - z_2) = 0,$$

or $B = 0$, contrary to hypothesis.

In the second case we have, $g(z_1, z_1) - g(z_2, z_2) =$

$$A(z_1 - z_2) + B(z_1 - z_2) = (A+B)(z_1 - z_2) = 0$$

$$g(z_1, z_2) - g(z_2, z_1) = A(z_1 - z_2) - B(z_1 - z_2) =$$

$$(A-B)(z_1 - z_2) = 0.$$

Therefore, since z_1 and z_2 are distinct $A+B = A-B = 0$ and

$A = O, B = 0$. Q.E.D.

6. Representations of the 7 Second Order Algebras by Inhomogeneous Quadratic Forms

All of the second order composition algebras can be represented by families of inhomogeneous quadratic forms. In addition, there are specific surfaces which can hold two distinct second order algebras. These objects generalize, as was stated at the beginning , like the fixed and periodic points of

functions of a single variable. We will show why the collection of second order algebras on a particular surface must be either all sensitive or all (left or right) insensitive. There exist surfaces on which all the insensitive algebras are found trivially. Simple combinatorial arguments will show why no second degree, or quadric, surface can hold 3 sensitive algebras.

Write the general equation of a quadratic function of two variables as

$$f(x,y) = ax^2 + by^2 + cxy + dx + ey + q.$$

We suppose that f represents a certain algebra K on at least two values z_1 and z_2, and that its table is given by f(z_i , z_j) = z_{ij} ε Z_2 . This statement is equivalent to this set of four equations:

$$(\alpha):(a+b+c)z_1^2 + (d+e)z_1 + q = z_{11}$$

$$(\beta):az_1^2 + bz_2^2 + cz_1z_2 + dz_1 + ez_2 + q = z_{12}$$

$$(\gamma):az_2^2 + bz_1^2 + cz_1z_2 + dz_2 + ez_1 + q = z_{21}$$

$$(\delta):(a+b+c)z_2^2 + (d+e)z_2 + q = z_{22}$$

Define the constants:

k = z_1 + z_2,

h = a+b+c ,

l = e+d.

The case h = 0 will be called *degenerate* for reasons that will be evident later. We will need 3 structure constants, given by :

$$\varepsilon_1 = (z_{11} - z_{12} / (z_1 - z_2)$$

$$\varepsilon_2 = (z_{11} - z_{21}) / (z_1 - z_2)$$

$$\varepsilon_3 = (z_{11} - z_{22}) / (z_1 - z_2)$$

We adopt the following convention: If an algebra K has only a single fixed point, that point is z_1. In combination with this convention, the structure constants completely determine the algebra.

The number of fixed points is given by ε_3 :

(a) $\varepsilon_3 = 1$, both points are fixed. Then ε_1, ε_2 can only assume values 0 or 1. If

(b) $\varepsilon_3 = 0$, there is only one fixed point, which by our convention is z_1. Once again ε_1, ε_2 can only assume values 0 or 1. If

(c) $\varepsilon_3 = -1$, then neither point is fixed: ε_1, ε_2 can only assume values 0 or -1.

The equations (α) and (β) will be called the outer set of equations. (γ) and (δ) are the inner set.

THEOREM II: If z= f(x,y) is an inhomogeneous quadratic form in two variables , then there exist at most 6 values the variables (x,y,z), , either real or complex, which can be combined in pairs to produce a second order composition algebra on its surface. These, furthermore, can be combined in only four ways These values are:

$$
\begin{aligned}
s_1 &= (1 - l + \sqrt{(1-l)^2 - 4qh})/2h \\
s_2 &= (1 - l - \sqrt{(1-l)^2 - 4qh})/2h \\
s_3 &= (-1 - l + \sqrt{(1-l)^2 - 4qh})/2h \\
s_4 &= (-1 - l - \sqrt{(1-l)^2 - 4qh})/2h \\
s_5 &= (-1 - l + \sqrt{(1-l)^2 - 4qh - 4})/2h \\
s_6 &= (-1 - l + \sqrt{(1-l)^2 - 4qh - 4})/2h
\end{aligned}
$$

These expressions come from the outer set of equations. The four possible combinations for composition algebras are:

$K_1 = (s_1 , s_2)$

$K_2 = (s_1, s_4)$

$K_3 = (s_2 , s_3)$

$K_4 = (s_5 , s_6)$

PROOF: Subtracting (δ) from (α) gives :

#25...

$$h(z_1^2 - z_2^2) + l(z_1 - z_2) = z_{11} - z_{22}$$
$$= \varepsilon_3(z_1 - z_2) \neq 0$$

Factoring out $z_1 - z_2$ **:**

$$h(z_1 + z_2) + l = hk + l = \varepsilon_3$$
$$k = (\varepsilon_3 - l) / h$$

Case I: $\varepsilon_3 = 1$ **. This is the condition for a second -order algebra of two fixed points. They will therefore be the two roots of the equation:**

$$hs^2 + ls + q = s$$
$$s_1, s_2 = (1 - l \pm \sqrt{(1-l)^2 - 4fh}) / 2h$$

Case II: $\varepsilon_3 = 0$ **. By the convention adopted, this signifies an algebra in which z_1 is fixed and z_2 is not. Thus. z_1 can be any one of the two values s_1 or s_2 , while z_2 can be s_3 or s_4 , which are solutions of:**

$$hs^2 + ls + q = k - s = ((\varepsilon_3 - l) / h) - s$$
$$= -l / h$$
$$s_3, s_4 = (-1 - l \pm \sqrt{(1-l)^2 - 4fh}) / 2h$$

Since $z_1 + z_2 = k = -(l /h)$, these must combine in the pairs (s_1 ,s_4) , or (s_2 , s_3) in any algebra of this type.

Case III : $\varepsilon_3 = -1$**. There are no fixed points. Both z_1 and z_2 are the twin solutions of the same equation, which is:**

$$hs^2 + ls + q = k - s = (\varepsilon_3 - l) / h - s$$
$$= -(1 + l) / h - s$$
$$s_5, s_6 = (-1 - l \pm \sqrt{(1+l)^2 - 4h(q + (1+l) / h)}) / 2h$$
$$= (-(1 + l) \pm \sqrt{(1-l)^2 - 4qh - 4}) / 2h$$

The 4 possible algebras on the surface determined by f(x,y) are therefore (s_1 , s_2) , (s_1 , s_4) , (s_2 , s_3) , and (s_5 , s_6) . Q.E.D.

We will describe the conditions on the 6 coefficients of f that determine the presence of one or more of the seven basic second order algebras. By subtracting (α) , (β) , (γ) ,(δ) from each other and dividing through by z1 - z2 , one obtains this set of 6 equations:

$$(i) bk + cz_1 + e = \varepsilon_1$$
$$(ii) bk + cz_2 + e = \varepsilon_3 - \varepsilon_2$$
$$(iii) ak + cz_1 + d = \varepsilon_2$$
$$(iv) ak + cz_2 + d = \varepsilon_3 - \varepsilon_1$$
$$(v) hk + l = \varepsilon_3$$
$$(vi) (a-b)k + d - e = \varepsilon_2 - \varepsilon_1$$

Subtracting (ii) from (i) gives the important relationship:

$c(z_1 - z_2) = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$ **Let**

$\delta = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$. **Then**

$$\underline{c = \delta / (z_1 - z_2)}$$

c is uniquely determined by the values of the z's and the structure constants of the algebra. *This is not true of any of the other coefficients.* However, since the above set of 6 relationships is not independent, we can let two of the coefficients be arbitrary, expressing the rest in terms of them, the specific values of the z's, and the structure constants.

These coefficients can be derived from the outer set of tabular equations (α) and (δ):

$$hz_1^2 + lz_1 + q = z_{11}$$

$$hz_2^2 + lz_2 + q = z_{22}$$

Transfer q to the right side and treat these as simultaneous equations in the unknowns h and l. Solving for h :

$$h = \begin{vmatrix} z_{11} - q & z_1 \\ z_{22} - q & z_2 \end{vmatrix} / \begin{vmatrix} z_1^2 & z_1 \\ z_2^2 & z_2 \end{vmatrix}$$

$$= ((z_{11}z_2 - z_{22}z_1) + q(z_1 - z_2)) / (z_1^2 z_2 - z_2^2 z_1)$$

$$= ((z_{11}z_2 - z_{22}z_1) / z_1 z_2 (z_1 - z_2)) + q / z_1 z_2$$

Let

$$D = ((z_{11}z_2 - z_{22}z_1) / z_1 z_2 (z_1 - z_2))$$

Then,

$$\underline{\underline{a + b + c = D + q / z_1 z_2}}$$

This is the fundamental equation linking the coefficients of the quadratic form f to the algebra K. "c "and "D" can be computed directly from the structure constants of the algebra and the specifications z_1 and z_2 . Selecting "a "and "q" as free parameters, "b" can then be derived from the above equation. Given these quantities, we can now solve for "d" and "e" from equations (i) and (iii) :

$$e = \varepsilon_1 - cz_1 - bk$$
$$d = \varepsilon_2 - cz_1 - ak$$

THEOREM III: Given the second order algebra K, the specifications z_1 and z_2 , and letting a and q function as free parameters, one can derive all of the coefficients of the quadratic forms f (x,y) representing K , from the equations:

$$E_1: \quad a + b + c = D + q / z_1 z_2$$
$$E_2: \quad c = \delta / (z_1 - z_2); \delta = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$$
$$E_3: \quad D = (z_{11} z_2 - z_{22} z_1) / (z_1 z_2 (z_1 - z_2))$$
$$E_4: \quad e = \varepsilon_1 - cz_1 - bk$$
$$E_5: \quad d = \varepsilon_2 - cz_1 - ak$$
$$E_6: \quad k = z_1 + z_2$$

Before listing the 7 algebras with their polynomial representations, observe that it is possible to rewrite the polynomial expression for f as a sum of three quadratics, two of which are the same in all representations and vanish identically on z_1 and z_2 . Substituting for b, c, etc., one has:

$$f(x,y) = ax^2 + by^2 + cxy + dx + ey + q$$
$$= ax^2 + (D + q / z_1 z_2 - c - a)y^2 + cxy + (\varepsilon_2 - cz_1 - ak)x$$
$$+(ak - kq / z_1 z_2 + \varepsilon_1 + c(k - z_1) - kD)y + q$$
$$= aP(x,y) + qQ(x,y) + R(x,y), where$$
$$P = a(x^2 - y^2 - k(x - y))$$
$$Q = q(y^2 / z_1 z_2 - ky / z_1 z_2 + 1)$$
$$R = y^2(D - c) + cxy + (\varepsilon_2 - cz_1)x + (\varepsilon_1 + cz_2 - kD)y$$

P(x,y) and Q(x,y) are the same for all algebras, and for most purposes superfluous. The catalog presented below gives K, its structure constants, D , and R(x,y):

CATALOG A

Algebra I:

$$\wedge (Boolean\text{"}and\text{"})$$

$K =$

y / z	z_1	z_2
z_1	z_1	z_1
z_2	z_1	z_2

$$\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = 1, \delta = -1, D = 0$$

$$R = (y^2 - xy + z_1 x - z_2 y) / (z_1 - z_2)$$

Algebra II:

L_x

$K =$

y / z	z_1	z_2
z_1	z_1	z_1
z_2	z_2	z_2

$$\varepsilon_1 = 0, \varepsilon_2 = 1, \varepsilon_3 = 1, \delta = 0, D = 0$$

$$R = x$$

Algebra III:

#30...

$$O_{z_1}$$

$$K = \begin{array}{c|c|c} y/z & z_1 & z_2 \\ \hline z_1 & z_1 & z_1 \\ \hline z_2 & z_1 & z_1 \end{array}$$

$$\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = 0, \delta = 0, D = -1/z_2$$

$$R = (ky - y^2)/z_2$$

Algebra IV:

$$G_{z_1}$$

$$K = \begin{array}{c|c|c} y/z & z_1 & z_2 \\ \hline z_1 & z_1 & z_2 \\ \hline z_2 & z_2 & z_1 \end{array}$$

$$\varepsilon_1 = 1, \varepsilon_2 = 1, \varepsilon_3 = 0, \delta = 2, D = -1/z_2$$

$$R = \frac{-ky^2 + 2z_2x(y-k) + kz_1y}{z_2(z_1 - z_2)}$$

#31...

Algebra V:

R4

$K =$

y \ z	z_1	z_2
z_1	z_1	z_1
z_2	z_2	z_1

$$\varepsilon_1 = 0, \varepsilon_2 = 1, \varepsilon_3 = 0, \delta = 1, D = -1 / z_2$$

$$R = \frac{-z_1 y^2 + z_2 xy - z_2{}^2 x + z_1{}^2 y}{z_2(z_1 - z_2)}$$

Algebra VI:

$$X_{z_2}$$

$K =$

y \ z	z_1	z_2
z_1	z_2	z_2
z_2	z_2	z_1

$$\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -1, \delta = 1$$

$$D = \frac{-(z_1 + z_2)}{z_1 z_2}$$

$$R = \frac{((z_2^2 - z_1 z_2 - z_1^2)y^2 + z_1 z_2 x((z_1 - z_2)y - z_1)}{z_1 z_2 (z_1 - z_2)}$$

$$+ \frac{k^2 y}{z_1 z_2}$$

Algebra VII:

Bx

$$K = \begin{array}{c|c|c} y/z & z_1 & z_2 \\ \hline z_1 & z_2 & z_2 \\ \hline z_2 & z_1 & z_1 \end{array}$$

$$\varepsilon_1 = 0, \varepsilon_2 = -1, \varepsilon_3 = -1, \delta = 0,$$
$$D = -(z_1 + z_2)/(z_2 z_1)$$
$$R = \frac{-ky^2 - z_1 z_2 x + k^2 y}{z_1 z_2}$$

****
****

7. Surfaces With Two Distinct 2nd Order Algebras

Having characterized all the quadric surfaces holding second order composition algebras, we now investigate the conditions under which such surfaces may support two different algebras. It was shown that at most four algebras are possible :

$K_1 = (s_1, s_2)$

$K_2 = (s_1, s_4)$

$K_3 = (s_2, s_3)$

$K_4 = (s_5, s_6)$

Symmetry arguments show that we need only examine four possibilities. S holds either

I: K_1, K_2

II: K_2, K_3

III: K_2, K_4, or

IV: K_1, K_4

CASE I: $K_1 = (s_1, s_2)$; $K_2 = (s_1, s_4)$

We distinguish between the structure constants of K_1 and K_2, and write:

K1: $\varepsilon^1_1, \varepsilon^1_2, \varepsilon^1_{3(=1)}$;

$$\delta_1 = \varepsilon^1_1 + \varepsilon^1_2 - \varepsilon^1_3$$

$$k_1 = s_1 + s_2 = (1-l)/h$$

K2: $\varepsilon^2_1, \varepsilon^2_2, \varepsilon^2_{3(=0)}$;

$$\delta_2 = \varepsilon^2_1 + \varepsilon^2_2 - \varepsilon^2_3$$

$$k_2 = s_1 + s_4 = -l/h$$

Let g= $\sqrt{(1-l)^2 - 4qh}$

From the equations derived in the previous section one has :

$$c = \delta_1 / (s_1 - s_2) = \delta_2 / (s_1 - s_4)$$

$$s_1 - s_2 = g/h$$

$$s_1 - s_4 = (1+g)/h \,.$$

Therefore $(1+g)\delta_1 = g\delta_2$

If *any one* of the three values c , δ_1 **,** δ_2 **are equal to 0, then, (as the** s_j **are assumed to be distinct), they *all* equal 0. Leaving this aside for the moment, we assume c ≠ 0. Solving for g gives**

$$g = \delta_1 / (\delta_2 - \delta_1)$$

Since g always exists and $\delta_1 \neq 0$**, it follows that** $\delta_1 \neq \delta_2$ **.**

The possibilities are therefore:

$\delta_1 = \varepsilon^1_1 + \varepsilon^1_2 - 1$	$\delta_2 = \varepsilon^2_1 + \varepsilon^2_2 - 0$
-1	1,2
+1	2

This translates into 3 options:

(i) g = -1/(1-(-1)) = -1/2 (δ_1 **=-1 ,** δ_2 **= +1)**

(ii) g = -1/(2-(-1)) = -1/3 (δ_1 **=-1** δ_2 **= +2)**

(iii) g = 1/(2-1) = +1 (δ_1 **=+1** δ_2 **= +2)**

Option (i) **: Substitute -1/2 for g in the expression for c to derive**

c = 2h. Since h = a+b+c , a+b = -h

$$s_1 = (1 - l + g) / 2h \ldots = (1 - 2l) / 4h$$

$$s_2 = (1 - l - g) / 2h \ldots = (3 - 2l) / 4h$$

$$s_4 = (-1 - l - g) / 2h \ldots = -(3 + 2l) / 4h$$

Under the assumption δ_1 =-1 , δ_2 = +1 one has $\varepsilon_1^1=0$, $\varepsilon_2^1=0$, and either $\varepsilon_1^2=1$, $\varepsilon_2^2=0$, or $\varepsilon_1^2=0$, $\varepsilon_2^2=1$.

These restrictions produce isomorphic algebras, so assume the former, namely $\varepsilon_1^2=1$, $\varepsilon_2^2=0$.

Going back to the fundamental set of relations (i)...(vi) (page 31) and writing $z_1 = s_1$, $z_2 = s_2$ for the first case, and $z_1=s_1$, $z_2 = s_4$ for the second case, we have:

$ak_1 + cs_1 + d = \varepsilon_2^1=0$,

$ak_2 + cs_1 + d = \varepsilon_2^2=0$. This implies a=0

$bk_1 + cs_1 + e = \varepsilon_1^1=0$,

$bk_2 + cs_1 + e = \varepsilon_1^2=1$. Therefore $b(k_2 - k_1) = 1$,

which works out to

a=0 , b = -h, c=2h

From these same equations:

$$d = -cs_1 = -2h((1 - 2l) / 4h)$$

$$= (2l - 1) / 2 = l - 1 / 2 = d + e - 1 / 2$$

$$\therefore e = 1 / 2$$

Lastly we can get q from

$$g = -1 / 2; (1 - l)^2 - 4qh = 1 / 4$$

$$= (1 / 2 - d)^2 - 4qh = 1 / 4 - d + d^2 - 4qh$$

$$\therefore q = (d^2 - d) / 4h$$

THEOREM IV : The quadric surfaces holding these two algebras:

$K_1(\wedge)$:

y / x	z_1	z_2
z_1	z_1	z_1
z_2	z_1	z_2

,

and

$K_2(R_3)$:

y / x	z_1	z_2
z_1	z_1	z_2
z_2	z_1	z_1

are given by the family of equations:

$$z = f(x, y) = -hy^2 + 2hxy + dx + (1/2)y + (d^2 - d)/4h$$

Option (ii) : $g = -1/(2-(-1)) = -1/3$ $\delta_1 = -1$, $\delta_2 = +2$

$\varepsilon^1_1 = \varepsilon^1_2 = 0$, $\varepsilon^2_1 = \varepsilon^2_2 = 1$,

$$ak_1 + c\,s_1 + d = \varepsilon^1_2 = 0,$$
$$ak_2 + c\,s_1 + d = \varepsilon^2_2 = 1.$$

$$bk_1 + cs_1 + e = \varepsilon^1_1 = 0,$$
$$bk_2 + cs_1 + e = \varepsilon^2_1 = 1.$$

$a(k_1 - k_2) = -1 = b(k_1 - k_2)$, so that $a = b$ (unless $h = 0$, which is treated separately) . Since $k_1 - k_2 = 1/h$, $a = -h$. Therefore

$$a = -h,\ b = -h,\ c = 3h.$$

Then since

$$s_1 = (1 - l + g) / 2h ... = (2 - l) / 2h$$
, we

see that

$$0 = ak_1 + cs_1 + d = -h(1 - l) / h + 3h(2 - l) / 2h + d$$
$$= l - 1 + 3 / 2(2 - l) + d = l - 1 + 3 - 3 / 2l + d$$
$$= 2 + d - l / 2$$
$$\therefore d = l / 2 - 2$$

But one also has that

$$(a - b)k_1 + (d - e) = 0 = d - e;$$
$$d = e, d + e = l, d = (d + e) / 2 - 2 =$$
$$2d / 2 - 2 = d - 2$$
$$\therefore 2 = 0!$$

THEOREM V : The following combination of algebras *will not be found* on any quadric surface in 3-space:

$K_1(\wedge)$:

$y \diagup x$	z_1	z_2
z_1	z_1	z_1
z_2	z_1	z_2

,

and

$K_2(G_{z_1})$:

$y \diagup x$	z_1	z_2
z_1	z_1	z_2
z_2	z_2	z_1

Option (iii) **: g = 1/(2-1) = +1** δ_1 **=+1** δ_2 **= +2**

$\varepsilon_1^1 = \varepsilon_2^1 = 1$, $\varepsilon_1^2 = \varepsilon_2^2 = 1$

$$s_1 = (1 - l + g)/2h ... = (2 - l)/2h$$
$$s_2 = (1 - l - g)/2h ... = -l/2h$$
$$s_4 = (-1 - l - g)/2h ... = -(2 + l)/2h$$
$$s_1 - s_2 = 1/h, c = (\delta_1 h/g) = h = a + b + c$$
$$\therefore a + b = 0$$

But we can easily show that a=b, since

$$ak_1 + cs_1 + d = \varepsilon_2^1 = 1,$$
$$ak_2 + cs_1 + d = \varepsilon_2^2 = 1.$$

$$bk_1 + cs_1 + e = \varepsilon_1^1 = 1$$
$$bk_2 + cs_1 + e = \varepsilon_1^2 = 1.$$

Since $k_1 \neq k_2$, both a=b=0 and d=e. From g = 1 we obtain

$$(1 - l)^2 - 4qh = 1; l = 2d; q = 3/4h$$

THEOREM VI : The pair of algebras:

$K_1(v)$:

y/x	z_1	z_2
z_1	z_1	z_2
z_2	z_2	z_2

$K_2(G_{z_1})$:

y/x	z_1	z_2
z_1	z_1	z_2
z_2	z_2	z_1

exist together on the family of surfaces in 3-space given by:

$$z = f(x, y) = hxy + d(x + y) + (d^2 - d)/4h$$

Finally we look the situation: c=0. This is trivially satisfied by all surfaces with no x variable : $z=hy^2+ly+q$.

Case II : $K_1 = (s_1, s_2)$ $K_2 = (s_5, s_6)$. Once again we first assume that $c \neq 0$.

Then $c = \delta_1/(s_1 - s_2) = \delta_2/(s_5 - s_6)$. Write,

$$g = \sqrt{(1-l)^2 - 4qh}$$

$$\sqrt{g^2 - 4} = \sqrt{(1-l)^2 - 4qh - 4}$$

For a pair of algebras with these traces one has:

$$\varepsilon_3^1 = 1, \varepsilon_3^2 = -1$$

$$\delta_1, \delta_2 \varepsilon \{-1, 0, 1\}$$

As the case c=0 is excluded for the moment, this means that

$$\delta_1 = \pm\delta_2 = \pm 1$$

$$1 / g = \pm 1 / \sqrt{g^2 - 4}$$

$$g^2 = g^2 - 4$$

$$4 = 0!$$

Therefore one cannot have two algebras of this kind on a surface unless c=0. **It turns out that in this instance, the solution for c=0 is non-trivial. Observe in passing that if c=0 , all of the algebras on the surface must be either right or left insensitive. The simplest way to show this is to try all combinations of ε's for δ=0 and note that they all produce insensitive algebras. However, we will now exhibit a family of surfaces which contains *both* a right *and* a left insensitive algebra.**

Let

$$\varepsilon_1^1 = 1, \varepsilon_2^1 = 0, \varepsilon_3^1 = 1$$

$$\varepsilon_1^2 = 0, \varepsilon_2^2 = -1, \varepsilon_3^2 = -1$$

$$\delta_1 = \delta_2 = 0$$

Using familiar procedures one has :

$$e = 0 - bk_1 = -1 - bk_2$$
$$d = 0 - ak_1 = -1 - ak_2, or:$$
$$b(k_1 - k_2) = +1 = a(k_1 - k_2).$$
$$\therefore a = b; c = 0 \rightarrow a = b = h/2$$
$$d = -ak_1 = -h/2((1-l)/h) = (l-1)/2$$
$$= (d + e - 1)/2$$
$$\therefore e = d + 1$$

THEOREM VII : **The family of surfaces defined by the equations:**

$$z = f(x,y) = a(x^2 + y^2) + d(x+y) + y + q$$

hold the two algebras:

$L_x =$

y/x	z_1	z_2
z_1	z_1	z_1
z_2	z_2	z_2

$L_y =$

y/x	z_1	z_2
z_1	z_1	z_2
z_2	z_1	z_2

Case III: $\mathbf{K_1} = (s_1, s_4)$ $\mathbf{K_2} = (s_2, s_3)$; $\varepsilon_3^1 = 0$ $\varepsilon_3^2 = 0$

#40...

$$s_1 - s_4 = (1+g)/h$$

$$s_2 - s_3 = (1-g)/h$$

$$g = \sqrt{(1-l)^2 - 4qh}$$

Therefore $c = \delta_1/(1+g) = \delta_2/(1-g)$

The situation g= 0 is excluded as it implies a double point. Therefore: $\delta_1 \neq \delta_2$. We also cannot have g ±1, since this also implies double points.

Solving the above equation gives:

$$\delta_1 - \delta_2 g = \delta_1 + \delta_2 g, or$$

$$g(\delta_1 + \delta_2) = \delta_1 - \delta_2$$

Assuming always that c ≠ 0 the possible values for the deltas are:

$$\delta_1 = \varepsilon_1^1 + \varepsilon_2^1; \delta_2 = \varepsilon_1^2 + \varepsilon_2^2;$$

$$\therefore \delta_1 = 2, \delta_2 = 1;$$

$$or, \delta_1 = 1, \delta_2 = 2, Therefore$$

$$\underline{\underline{g = \pm 1/3}}$$

Therefore assume that, (the other case being isomorphic):

$$\varepsilon_1^1 = 1, \varepsilon_2^1 = 1, \varepsilon_3^1 = 0$$

$$\varepsilon_1^2 = 0, \varepsilon_2^2 = 1, \varepsilon_3^2 = 0$$

$$\delta_1 = 2, \delta_2 = 1$$

Referring to the basic set of relations one has:

$$k_1 = k_2 = -l/h$$

$$d = \varepsilon_2^1 - cs_1 - ak_1 = 1 - cs_1 - ak_1$$

$$d = \varepsilon_2^2 - cs_3 - ak_2 = 1 - cs_3 - ak_1$$

$$\therefore s_1 = s_3$$

This is ruled out because s_1 is a fixed point and s_3 is not. ***It follows that the simultaneous existence of two algebras of this type on a surface is incompatible with the condition $c \neq 0$.*** **If we let c = 0 , it will follow that all the epsilons and deltas will be zero!Then**

$d = -ak_1 = (al)/h$

$e = (bl)/h$

$h = a+b$

$l = d+e$. Since $k_1 = k_2 = k = -(l/h)$, the desired family of equations must be of the form:

$$z = f(x,y) = ax^2 + by^2 - akx - bky + q$$
$$= ax(x-k) + by(y-k) + q$$
$$f(s_1, s_4) = as_1(s_1 - k) + bs_4(s_4 - k) + q$$
$$= -as_1s_2 - bs_4s_3 + q$$
$$= as_2s_1 - bs_3s_4q = f(s_2, s_3)(!!)\text{But}$$
$$f(s_1, s_4)\varepsilon\{s_1, s_4\}, f(s_2, s_3)\varepsilon\{s_2, s_3\}$$

In other words, the tables of the two algebras cannot contain common elements. Combining this with the previous result for $c \neq 0$, gives

THEOREM VIII : This pair of algebras is incompatible with any surface!

IV: K1 = (s_1, s_4) K2 = (s_5, s_6) ; $\varepsilon_3^1 = 0 \quad \varepsilon_3^2 = -1$. **As usual we assume that**

$c \neq 0$, $\delta_1 \neq 0$, $\delta_2 \neq 0$.

$$s_1 - s_4 = (1+g)/h$$
$$s_5 - s_6 = \sqrt{g^2 - 4}/h$$
$$g = \sqrt{(1-l)^2 - 4qh}$$
$$c = \delta_1/(s_1 - s_4) = \delta_2/(s_5 - s_6)$$

Then $= (\varepsilon_1^1 + \varepsilon_2^1)/(1+g) = (\varepsilon_1^2 + \varepsilon_2^2 + 1)/\sqrt{g^2 - 4}$

$$\therefore \delta_1^2 (g^2 - 4) = \delta_2^2 (1+g)^2, and$$
$$\underline{\underline{g^2(\delta_1^2 - \delta_2^2) - 2\delta_2^2 g - (4\delta_1^2 + \delta_2^2) = 0}}$$

The only possibilities for the deltas are $\delta_1 = 1,2$, $\delta_2 = -1,1$. Therefore

$$\delta_1^2 = 1,4; \delta_2^2 = 1$$

Combining these conditions gives rise to two equations for g, with three distinct roots:

$$(1.) - (2g + 5) = 0$$
$$(2.) 3g^2 - 2g - 17 = 0$$
$$g_1 = -5/2;$$
$$g_2 = (1 + 2\sqrt{13})/2; g_3 = (1 - 2\sqrt{13})/2$$

The two roots g_2 , g_3 correspond to isomorphic algebras. We need therefore only consider g_1 and g_2 .

Option (i) : $g = -5/2$. Then

$$\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = -1$$
$$(a) \varepsilon_1^1 = 0, \varepsilon_2^1 = 1, or$$
$$(b) \varepsilon_1^1 = 1, \varepsilon_2^1 = 0$$
$$\delta_1 = 0, \delta_2 = -1$$

(a) and (b) are equivalent so we choose (a) . Finally we can write down these basic relations:

#43...

$$d = 1 - cs_1 - ak_1 = 1 - cs_5 - ak_2$$

$$e = 0 - cs_1 - bk_1 = -1 - cs_5 - bk_2$$

$$k_1 = -l / h; k_2 = -(1 + l) / h$$

$$c(s_1 - s_5) + a(k_1 - k_2) = 2$$

$$c(s_1 - s_5) + b(k_1 - k_2) = 1$$

$$(a - b)(k_1 - k_2) = 1$$

$$= (a - b)(-l / h + (1 + l) / h) = (a - b)h$$

$$\therefore a - b = h = a + b + c, or$$

$$2b + c = 0;$$

$$c = \delta_1 h / (s_1 - s_4) = -2 / 3h, and$$

$$therefore:$$

$$\underline{\underline{a = 4b; c = -2b; h = 3b}}$$

We use the top equations to compute the values of c and d:

$$d - e = (b - a)k_2 = (b - 4b)(-1 - l)/3b$$
$$= 1 + l = d + e + 1, or$$
$$2e + 1 = 0; \therefore e = -1/2$$
$$d = 1 - cs_1 - ak_1 = 1 + 2b(3 + 2l)/12b + 4bl/3b$$
$$= 1 + (2l + 3)/6 + 4l/3 = 3/2 + 5(d + e)/3$$
$$= 3/2 + 5/3(d - 1/2) = 3/2 - 5/6 + 5/3d$$
$$\therefore d = -1(!)$$

A final surprise awaits us. Lets us find the value of q from the expression for g:

$$g = \sqrt{(1 - l)^2 - 4qh} = -5/2$$
$$l = d + e = -1/2 - 1 = -3/2$$
$$(1 + 3/2)^2 - 4qh = (5/2)^2 - 4qh = (5/2)^2$$
$$\therefore 4qh = 0 \Rightarrow q = 0(!)$$

,assuming as always that, for the moment h≠0.

THEOREM IX: **These two algebras:**

#45...

$$X_{z_1} = \begin{array}{c|c|c} y/x & z_1 & z_2 \\ \hline z_1 & z_2 & z_1 \\ \hline z_2 & z_1 & z_1 \end{array}$$

$$R_2 = \begin{array}{c|c|c} y/x & z_1 & z_2 \\ \hline z_1 & z_1 & z_1 \\ \hline z_2 & z_2 & z_1 \end{array}$$

can be found together on the family of surfaces given by

$$z = 4bx^2 + by^2 - 2bxy - x - 1/2y$$

$$s_1 = 0, s_4 = (1/2)b, s_5 = (-1/6)b, s_6 = (1/3)b$$

Option (ii) : $g = (1 + 2\sqrt{13})/2$

This option is by far the most exotic . We have

$$\delta_1 = \varepsilon_1^1 + \varepsilon_2^1 = 2; \delta_2 = \varepsilon_1^2 + \varepsilon_2^2 + 1 = \pm 1$$
$$\varepsilon_1^1 = \varepsilon_2^1 = 1, and$$
$$(a)\varepsilon_1^2 = \varepsilon_2^2 = -1; or$$
$$(b)\varepsilon_1^2 = \varepsilon_2^2 = 0$$

It turns out that cases (a) and (b) are equivalent, so we use case (a). Then:

$$c = \delta_1 h/(1+g) = 2h/(1+(1+2\sqrt{13}))/3)$$

$$= 6h/(4+2\sqrt{13}) = 3h/(2+\sqrt{13})$$

$$= h(\sqrt{13}-2)/3;$$

$$d = 1 - cs_1 - ak_1 = -cs_5 - ak_2$$

$$e = 1 - cs_1 - bk_1 = -cs_5 - bk_2;$$

$$(d-e) = (b-a)k_1 = (b-a)k_2$$

$$Since, k_1 = -l/h \neq k_2 = (-1-l)/h$$

$$\therefore a = b, d = e(!)$$

With the above set of equations we can now calculate

a (=b) in terms of either h or c:

$$h = 3c/(\sqrt{13}-2) = c(2+\sqrt{13})/3$$
$$= a+b+c = 2a+c$$
$$\therefore a = c(\sqrt{13}-1)/6 = b$$

THEOREM X : The two algebras:

$G_{z_1} =$

y/x	z_1	z_2
z_1	z_1	z_2
z_2	z_2	z_1

$X_{z2} =$

y/x	z_1	z_2
z_1	z_2	z_1
z_2	z_1	z_1

can be found together on the family of surfaces given by

$$z = \frac{(c(x^2+y^2)(\sqrt{13}-1))}{6} + cxy + d(x+y) + q$$

where the value of q may be obtained from the solution to the equation:

$$\sqrt{(1-2d)^2 - 4qc(2+\sqrt{13})/3}$$
$$= (1+2\sqrt{13})/3$$

We wrap up this analysis of the situation by considering the exceptional cases, c=0 , and h=0. For every situation in which c = 0 (except the one described in Case II) , either the x or the y variable drops out and we are dealing with single variable dynamics on insensitive algebras. In the degenerate situation, h = 0, the outer equations collapse; there can be at most one fixed point given by p = q/(1-l) = s_1 . There can't be any other composition algebra point since if there was such a point p*, it would have to satisfy f(p*,p*)= p ; but only p satisfies this.

However, if h = 0 and l=1, then every value x is a fixed point, as in the equation f(x,y) = x^2 - y^2 + x. For this equation, any pair of values (v,-v), will form an algebra that lives on its surface. The degenerate surfaces do have some interesting properties ,but we will not concern ourselves with them here.

CATALOG B

Quadric Surfaces Holding Two Distinct Second-Order Binary Composition Algebras

I.

$$K_1 = \wedge; \varepsilon_1^1 = 0, \varepsilon_2^1 = 0, \varepsilon_3^1 = 1$$

$$K_2 = R_3; \varepsilon_1^2 = 1, \varepsilon_2^2 = 0, \varepsilon_3^2 = 0$$

$$F(x,y) = -hy^2 + 2hxy + dx + y/2 + (d^2 - d)/4h$$

II.

$$K_1 = \wedge; \varepsilon_1^1 = 0, \varepsilon_2^1 = 0, \varepsilon_3^1 = 1$$

$$K_2 = G_{S_1}; \varepsilon_1^2 = 1, \varepsilon_2^2 = 1, \varepsilon_3^2 = 0$$

No Solution

III.

$$K_1 = \vee; \varepsilon_1^1 = 1, \varepsilon_2^1 = 1, \varepsilon_3^1 = 1$$

$$K_2 = G_{S_1}; \varepsilon_1^2 = 1, \varepsilon_2^2 = 1, \varepsilon_3^2 = 0$$

$$F(x,y) = hxy + d(x+y) + (d^2 - d)/h$$

IV:

$$K_1 : Either$$

$$\varepsilon_1^1 = 1, \varepsilon_2^1 = 0, \varepsilon_3^1 = 1, or$$

$$\varepsilon_1^1 = 1, \varepsilon_2^1 = 1, \varepsilon_3^1 = 1$$

$$K_2 : Either$$

$$\varepsilon_1^2 = 0, \varepsilon_2^2 = 0, \varepsilon_3^2 = -1, or$$

$$\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = -1$$

All impossible.

V.

$$K_1 = L_y : \varepsilon_1^1 = 1, \varepsilon_2^1 = 0, \varepsilon_3^1 = 1$$

$$K_2 = X_x : \varepsilon_1^1 = 1, \varepsilon_2^1 = -1, \varepsilon_3^1 = -1$$

$$F(x,y) = a(x^2 + y^2) + dx + (1+d)y + q$$

VI.

$$K_1 : \varepsilon_3^1 = 0$$

$$K_2 : \varepsilon_3^2 = 0$$

Two distinct algebras with 'constant trace'. Impossible if both algebras are sensitive (2-dimensional) . An insensitive algebra with a constant trace must be a constant algebra Such pairs have also been shown not to exist on any quadric surface.

#49...

VII:

$$K_1 = R_3 : \varepsilon_1^1 = 0, \varepsilon_2^1 = 1, \varepsilon_3^1 = 0$$

$$K_2 = X_x : \varepsilon_1^1 = \varepsilon_2^1 = \varepsilon_3^1 = -1$$

$$F(x,y) = 4bx^2 + by^2 - 2bxy - x - y/2$$

VIII:

$$K_1 = G_{s_1} : \varepsilon_1^1 = 1, \varepsilon_2^1 = 1, \varepsilon_3^1 = 0$$

$$K_2 = X_x : \varepsilon_1^1 = \varepsilon_2^1 = \varepsilon_3^1 = -1$$

$$F(x,y) = (-1 + \sqrt{13})c/6)(x^2 + y^2) + cxy + d(x + y) + q, where$$

$$q = (1/4h)((1 - 2d)^2 - (1 + 2\sqrt{13}/3)^2)$$

❋❋❋❋❋❋❋❋❋❋❋❋
❋❋❋❋❋❋❋❋❋❋❋❋

PART II

1. Dynamical Properties of 2nd Order Composition Algebras on Quadric Surfaces.

With the information presented in the preceding sections one can begin to describe the simplest features of the dynamics of algebras on surfaces. As usual, we assume initially that h≠0 , c≠0. Let the equation of a surface S be given by:

$$F(x,y) = ax^2 + by^2 + cxy + dx + ey + q$$

The coefficients will be such that one or more of binary composition algebras can be found on this surface. Let one of these be K , characterized by its' set of structure constants,

$$\varepsilon_1, \varepsilon_2, \varepsilon_3; \delta = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$$

The attractive or repelling behavior of the iterate collection of functions, Ψ_f , at a specific composition point p=(z_α, z_β , z_γ) , with z_γ =f (z_α , z_β) , is determined by the first and second derivatives F_x , F_y at that point. Since we are iterating around algebras and not on specific real numbers, an important generalization to the notion of convergence in a single variable will be introduced: that of the *ε-stable neighborhood of a point on the (x,y) plane.*

***Definition* . Let p = F(r,t) be given. We will say that p is ε-stable, or there is an ε-stable neighborhood around p if there exists a number ε such that**

$$|\alpha_1|, |\beta_1|, |\alpha_2|, |\beta_2| \le \varepsilon \rightarrow$$
$$|F(r+\alpha_1, t+\beta_1) - F(r+\alpha_2, t+\beta_2)| < \varepsilon$$

Notice the *relative* inequality for the increments α and β , and the *strict* inequality on the increment for F. One sees immediately the purpose of this definition in the case that (i) s =r=t, that is, s is a fixed point, and

(ii) $|F_x| + |F_y| < 1$. Given these conditions, the neighborhood of s will be ε-stable, and *every value q of every perturbed iterate of F of the form*

q= F(π_K (s+ε , s)) *will remain within the interval*

(s+ε,s-ε) *the epsilon-neighborhood around p.*

Another way of looking is that is to observe that if h=F(m,n) is any other point, and u= F(π_k(m,n)) is some iterate which happens to fall within an ε-neighborhood of s , then all iterates of F on u and s, of the form F(π_k (u,s)), will remain in that epsilon -neighborhood. One can therefore replace the idea of convergence of a point to a fixed point, or of a point to a periodic set of points, by that of an iterate clone being trapped within the neighborhood of a fixed point, or of an algebra of points, or of a sub-algebra of points. Indeed, we will be interested in *investigating the presence of ε-stable sub-algebras within a given composition algebra on a surface S.* (In the case in which the order is 2 of course , all the proper sub-algebras are the fixed points).

Definitions : Let S be a surface in 3-space define by an equation of the form z= F(x,y), generally a polynomial. Let there be an n^{th} -order composition algebra **K** = (Z, °) , Z=(z_1,z_2 ,z_3 ,.......z_n). on S, and let P = (z_i, z_j ,F(z_i, z_j)= z_{ij} ε Z) be a typical composition point on S. Then we will say that

(1) P is an *attractive point* if $|F_x| + |F_y| < 1, (x,y) = (z_\alpha, z_\beta)$

(2) P is *left, (right) , attractive* if $|F_x| < 1$ ($|F_y| < 1$)

(3) P is *borderline* if $|F_x| + |F_y| = 1$

(4) P is *left, (right) , borderline* if $|F_x| = 1$ ($|F_y| = 1$)

(5) P is *controllable* if $|F_x| < 1$, $|F_y| < 1$ but $|F_x| + |F_y| > 1$. That is to say, when P is both left and right attractive, but not attractive or borderline.

(6) P is *superattractive* if it is a critical point, that is ,when $F_x = 0 = F_y$

(7) P is *repelling, repulsive or repellant*, when $|F_x| > 1, |F_y| > 1$

(8) P is *hyperbolic* when

$$|F_x| < 1, |F_y| > 1, or$$

$$|F_x| > 1, |F_y| < 1$$

Using this collection of terms , we return to the tables of families of surfaces holding one or two second-order binary composition algebras, and identify the dynamical characteristics of each element in their tables.

After a general discussion , we will completely analyse the dynamics of the *first entry* in the table of the 7 families of surfaces, to show how the procedure is carried ou.The results for the other cases will then be listed .

IX. General Considerations

A given 2- algebra K will have 4 entries in its standard table, which become points on the surface S : $P_1=(z_1,z_1)$, $P_2=(z_1,z_2)$, $P_3=(z_2,z_1)$, $P_4=(z_2,z_2)$ **. The relations between coefficients of F, common to all such algebras, produce simple expressions for the derivatives at these points.**

P1:

$$F_x(z_1,z_1) = 2az_1 + cz_1 + d = 2az_1 + \varepsilon_2 - ak =$$
$$a(z_1 + z_1 - (z_1 + z_2)) + \varepsilon_2 = a(z_1 - z_2) + \varepsilon_2 .Since$$
$$c = \delta / (z_1 - z_2),$$
$$\therefore \underline{\underline{F_x(z_1,z_1) = \varepsilon_2 + a\delta / c}}. Likewise$$
$$\therefore F_y(z_1,z_1) = \varepsilon_1 + (b / c)\delta$$

P2 :

$$F_x(z_1,z_2) = 2az_1 + cz_2 + d = 2az_1 + \varepsilon_3 - \varepsilon_1 - ak =$$

$$\therefore F_x(z_1,z_2) = \varepsilon_3 - \varepsilon_1 + (a/c)\delta$$

$$F_y(z_1,z_2) = 2bz_2 + cz_1 + e =$$

$$2bz_2 + \varepsilon_1 - bk$$

$$\therefore F_y(z_1,z_2) = \varepsilon_1 - (b/c)\delta$$

P3 :

$$F_x(z_2,z_1) = 2az_2 + cz_1 + d = 2az_2 + \varepsilon_2 - ak$$

$$\therefore F_x(z_2,z_1) = \varepsilon_2 - (a/c)\delta. Likewise$$

$$\therefore F_y(z_2,z_1) = \varepsilon_3 - \varepsilon_2 + (b/c)\delta$$

P4 . Using the above methods, we obtain:

$$F_x(z_2,z_2) = \varepsilon_3 - \varepsilon_1 - (a/c)\delta. Likewise$$

$$\backslash F_y(z_2,z_2) = \varepsilon_3 - \varepsilon_2 - (b/c)\delta$$

THEOREM XI : The two derivatives at each point of a second-order binary composition algebra on a quadratic surface, are all simple expressions ,linear in the structure constants of the algebra, and in the ratios a/c and b/c.

2. Analysis of the Dynamics of the Algebra

$$K = \begin{array}{c|c|c} y/x & z_1 & z_2 \\ \hline z_1 & z_1 & z_1 \\ \hline z_2 & z_1 & z_2 \end{array} = \vee;$$

$$\varepsilon_1 = 0, \varepsilon_2 = 0 \, \varepsilon_3 = 1, \delta = -1, D = 0$$

I. F(x,y) = aP(x,y) +qQ(y) + R(x,y)=

$$a(x^2 - y^2 - k(x-y)) + q(1 + (y^2 - ky)/z_1z_2)) +$$

$$(y^2 - xy + z_1x - z_2y)/(z_1 - z_2)$$

For the purposes of describing the dynamical behavior in the

neighborhood of the composition points $P_1 ... P_4$, it will be convenient to take a,b,c and q as free parameters. Once these are chosen, then the values z_1, z_2 will be completely determined. From the above equations we see that:

$$\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = 1, \delta = -1, D = 0;$$
$$z_1 z_2 = q/(a+b+c) = (q/c)/(\mu+1)$$
$$z_2 - z_1 = 1/c. So$$
$$z_1(1/c + z_1) = q/(c(\mu+1)), and$$
$$\therefore z_1, z_2 = (-1/c \pm \sqrt{(1/c)^2 + 4q/(c(\mu+1))})/2 =$$
$$= (-1 \pm \sqrt{1 + 4qc/(\mu+1)})/2c$$

It will generally be true that the values z_1 ,z_2 depend only on a,b,c, q and the structure constants. Since the derivatives of the surface funtion at the composition points depend only on the ratios a/c, b/c, and the structure constants, we will not be interested in the specific values of the composition points P1 ... P4, but only in these terms.

For the algebra $K =$

$y \diagup x$	z_1	z_2
z_1	z_1	z_1
z_2	z_1	z_2

, therefore , one has,

$P_1: f(z_1, z_1) = z_1$ $f_x = \varepsilon_2 + \delta a/c = -a/c$ $f_y = \varepsilon_1 + \delta b/c = -b/c$	$P_2: f(z_1, z_2) = z_1$ $f_x = \varepsilon_3 - \varepsilon_1 + \delta a/c = 1 - a/c$ $f_y = \varepsilon_1 - \delta b/c = +b/c$
$P_3: f(z_1, z_2) = z_1$ $f_x = \varepsilon_2 - \delta a/c = +a/c$ $f_y = \varepsilon_3 - \varepsilon_2 + \delta b/c = 1 - b/c$	$P_4: f(z_1, z_2) = z_2$ $f_x = \varepsilon_3 - \varepsilon_1 - \delta a/c = 1 + a/c$ $f_y = \varepsilon_3 - \varepsilon_2 - \delta b/c = 1 + b/c$

For P_1 to be attractive, one must have $|a/c| + |b/c| < 1$. This limits the ranges in which P_2 and P_3 will be attractive. Under the assumption that a/c

is positive,<1, P_2 will be attractive when

$$1 - a/c + |b/c| < 1, or$$

$$1 > |a/c| + |b/c| > a/c > |b/c|$$

$$\therefore |b/c| < 1/2, and 1/2 < a/c < 1 - |b/c|$$

The dual situation pertains to P_3. Clearly we cannot have P_1 , P_2 and P_3 all attractive. We can however make them all borderline by letting a/c=b/c=+1/2. In that case P_4 will be repellant. If P_4 is attractive, then both a/c and b/c are negative and

$$1 > |1 + a/c| + |1 + b/c| = 1 - |a/c| + 1 - |b/c|$$

$$= 2 - (|a/c| + |b/c|);$$

$$\therefore |a/c| + |b/c| > 1$$

If P_4 is attractive, then P_1 is at most controllable, P_2 and P_3 are hyperbolic. We can begin to examine what I tentatively call "circuits". These are schemas of "transmission" of perturbations between several or all of the composition points of a surface algebra which remain within the ε-stable neighborhoods around these points.

Since those algebras for which c = δ = 0 require a different kind of analysis, we move next to Algebra IV on table I , that is to say

G_{z_1} =

y / x	z_1	z_2
z_1	z_1	z_2
z_2	z_2	z_1

$$\varepsilon_1 = 1, \varepsilon_2 = 1, \varepsilon_3 = 0, \delta = 2, D = -1/z_2$$

Following the procedures established for Algebra I, we have:

$P_1: f(z_1,z_1) = z_1$	$P_2: f(z_1,z_2) = z_2$
$f_x = 1 + 2a/c$	$f_x = -1 + 2a/c$
$f_y = 1 + 2b/c$	$f_y = 1 - 2b/c$
$P_3: f(z_2,z_1) = z_2$	$P_4: f(z_2,z_2) = z_2$
$f_x = 1 - 2a/c$	$f_x = -(1 + 2a/c)$
$f_y = -1 + 2b/c$	$f_y = -(1 + 2b/c)$

P_1 attractive --> a/c,b/c < 1. Then P_2 is repelling, P_3 is repelling, P4 is attractive. Since this is the group among the second order algebras , one can speak about the algebraic structure of “group dynamics [1]“ . Let us also look at the range $-1<a/c<0$ and $0<b/c<1$. In this case P1 is hyperbolic, left attractive; Ps is hyperbolic, right attractive; P3 is hyperbolic, left attractive and P4 is hyperbolic, left attractive.

Algebra V:

$R_4 =$

$y \diagup x$	z_1	z_2
z_1	z_1	z_1
z_2	z_2	z_1

$$\varepsilon_1 = 0, \varepsilon_2 = 1, \varepsilon_3 = 0, \delta = 1, D = -1/z_2$$

[1] **For those who might think otherwise, this is not a paper in social psychiatry!**

$P_1: f(z_1,z_1) = z_1$	$P_2: f(z_1,z_2) = z_1$
$f_x = 1 + a/c$	$f_x = -1 + a/c$
$f_y = 1 + b/c$	$f_y = -b/c$
$P_3: f(z_2,z_1) = z_2$	$P_4: f(z_2,z_2) = z_1$
$f_x = 1 - a/c$	$f_x = -(1 + a/c)$
$f_y = b/c$	$f_y = -b/c$

-1<b/c<+1 : In this range all four points are right attractive. When

-1<a/c<0, P1 and P2 are left attractive

0<a/c<1, then P3 and P4 are right attractive. Investigating the conditions under which a point is totally attractive we see that in the range -1<a/c<0 and -1<b/c<+1, we have

$$\left|f_x\right| + \left|f_y\right| = 1 + a/c + |b/c| < 1$$

$$\rightarrow a/c + |b/c| < 0, or$$

$$-1 < a/c < -|b/c|$$

In the range -1<b/c<+1 , and -2<a/c<-1, we have

$$\left|f_x\right| + \left|f_y\right| = -1 - a/c + |b/c| < 1$$

$$\rightarrow a/c + |b/c| < 2 \rightarrow |a/c| + |b/c| < 2.$$

It follows that the conditions for the existence of a pair of attractive points are:

$$|b/c| < 1; -2 < a/c < 0. Either$$

$$|a/c| > |b/c|, (-1 < a/c < 0), or$$

$$|a/c| + |b/c| < 2(-2 < a/c < -1$$

$$\therefore P_1, P_4 attractive \rightarrow P_2, P_3 hyperbolic$$

$$P_1, P_4 hyperbolic \rightarrow P_2, P_3 attractive$$

Algebra VI:

$$X_{z_2} = \begin{array}{c|c|c} y/x & z_1 & z_2 \\ \hline z_1 & z_2 & z_2 \\ \hline z_2 & z_2 & z_1 \end{array}$$

$$\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -1, \delta = 1, D = -(z_1 + z_2)/(z_2 - z_1)$$

$P_1: f(z_1, z_1) = z_2$	$P_2: f(z_1, z_2) = z_1$
$f_x = a/c$	$f_x = -1 + a/c$
$f_y = b/c$	$f_y = -b/c$
$P_3: f(z_2, z_1) = z_1$	$P_4: f(z_2, z_2) = z_2$
$f_x = -a/c$	$f_x = -(1 + a/c)$
$f_y = -1 + b/c$	$f_y = -(1 + b/c)$

0 < a/c , b/c < 1, with a/c + b/c <1 ----> P1 attractive----> P4 repelling ----> either P2 attractive and P3 controllable(b/c<a/c) ,or P2 controllable and P3 attractive(b/c > a/c).

This covers all of the algebras listed in Table I for which c ≠0. All second order composition algebras for which c = 0 are insensitive, or, shall we say, 1-dimensional. To all intents and purposes we are dealing with single-variable dynamics, a subject being studied with great enthusiasm . The particular features of 1-dimensional dynamics on a 2-surface are interesting, but we will not be looking at them here.

❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋

Finally we briefly examine the dynamics surrounding the composition points on surfaces holding 2 distinct 2nd-order composition algebras. I refer the reader to Catalog B:

I.

$K_1 = \wedge; \varepsilon_1^1 = 0, \varepsilon_2^1 = 0, \varepsilon_3^1 = 1$

$K_2 = R_3; \varepsilon_1^2 = 1, \varepsilon_2^2 = 0, \varepsilon_3^2 = 0$

$F(x,y) = -hy^2 + 2hxy + dx + y/2 + (d^2 - d)/4h$

$P_1: f(s_1, s_1) = s_1; f_x = 0; f_y = 1/2$

$P_2: f(s_1, s_2) = s_1; f_x = 1; f_y = -1/2$

$P_3: f(s_2, s_1) = s_1; f_x = 0; f_y = 3/2$

$P_4: f(s_2, s_2) = s_2; f_x = 1; f_y = 1/2$

$P_5: f(s_1, s_4) = s_4; f_x = 1; f_y = -1/2$

$P_6: f(s_4, s_1) = s_1; f_x = 0; f_y = -1/2$

$P_7: f(s_4, s_4) = s_1; f_x = -1; f_y = -1/2$

III.

$K_1 = \vee; \varepsilon_1^1 = 1, \varepsilon_2^1 = 1, \varepsilon_3^1 = 1$

$K_2 = G_{s_1}; \varepsilon_1^2 = 1, \varepsilon_2^2 = 1, \varepsilon_3^2 = 0$

$F(x,y) = hxy + d(x+y) + (d^2 - d)/h$

$P_1: f(s_1, s_1) = s_1; f_x = 1; f_y = 1$

$P_2: f(s_1, s_2) = s_2; f_x = 0; f_y = 1$

$P_3: f(s_2, s_1) = s_2; f_x = 1; f_y = 0$

$P_4: f(s_2, s_2) = s_2; f_x = 0; f_y = 0$

$P_5: f(s_1, s_4) = s_4; f_x = -1; f_y = 1$

$P_6: f(s_4, s_1) = s_1; f_x = 1; f_y = -1$

$P_7: f(s_4, s_4) = s_1; f_x = -1; f_y = -1$

V.

$$K_1 = L_y\!: \varepsilon_1^1 = 1, \varepsilon_2^1 = 0, \varepsilon_3^1 = 1$$

$$K_2 = X_x\!: \varepsilon_1^1 = 1, \varepsilon_2^1 = -1, \varepsilon_3^1 = -1$$

$$F(x,y) = a(x^2 + y^2) + dx + (1+d)y + q$$

Since both algebras are 1-dimensional, we will not examine this situation here. There are two more families of surfaces in Table II, but there is no need to describe their dynamical properties in detail. Like the others , the derivatives at each composition point can be expressed as real numbers , from which their dynamical properties are immediately apparent .

PART III

1. Summary of Parts I and II :

The point of departure for this investigation is in the observation that the natural extension of the periodic and fixed points of univariate dimensional dynamical systems to bivariate systems, are the finite n^{th}-order composition algebras, often referred to in the literature as "groupoids". We do not advocate this terminology as there is little that is group-like about these objects. An n^{th} order binary composition algebra K, can be represented by a set of indeterminates , $Z = \{z_1, z_2, \ldots\ldots, z_n\}$, closed under some binary operation, ∘ , with $z_i \circ z_j = z_{ij}$ __Z. Using the notation $K^n = \{Z, \circ\}$ this algebra can be "represented" by polynomials $z = P^{(m)}(x,y)$ of various degrees m , over the complex numbers, such that for some set or sets of specific values $\{z_i\}$ I = 1,2......n , one has $P(z_i, z_j) = z_{ij}\ \varepsilon\ Z$.

The extension of the iterate set of functions derived from a function , w = f(z), of a single variable will then be the *clone* of compositions of P with itself :

$\Psi_P = \{x, y, P(x,x), P(x,y), P(y,x), P(y,y), P(x,P(x,y))\ldots.\}$

$= \{P(\pi_0), P(\pi_1), P(\pi_2), \ldots\ldots\ldots\}$ In part I these clones are notated by the symbol Ψ_P, $\Pi(x,y)$. Symbolically,

$\Psi_P = P(\Pi$___

Ψ_P in turn gives rise to an associated clone of algebras, $\boxed{K^{n_0}, K^{n_1}, \ldots, K^{n_j},}$ that relate to Ψ_P like a residue class (mod P). The elements of Ψ_P can be enumerated via the Cantor *J* function; the description of the residue classes then becomes a problem in Diophantine equations .

The rest of part I is taken up with the detailed study of the representation of second-order composition algebras on quadric surfaces of the form $z = ax^2 + by^2 + cxy + dx + ey + q$. There are 7 such distinct algebras. We distinguish between 1-dimensional (right or left insensitive) , and 2-dimensional algebras, according to whether they are responsive to one or two variables. The

complete description of those families of quadratic surfaces holding each algebra is presented, as well as a catalog that lists those surfaces holding two or more algebras,

Finally, the dynamic profile at each composition point on each surface is computed, and the points are classified according to whether they are attractive, repellant, borderline, controllable or hyperbolic.

2. On the relationship of the order of a composition algebra to the degrees of its polynomial representations

Let $K^n = \{ Z, \circ \}$ be some finite binary composition algebra. The number n of elements in Z, will be called the order of the algebra. When K can be represented on a polynomial surface $z = P(x,y;m) = \sum_{0 \le i+j \le m}^{m} a_{ij} x^i y^j$, then the exponent m will be called the degree of the representation. If m is the smallest such exponent for the given algebra, then we can speak of m as being the "degree" of K itself.

***Example**: The following algebra will not be found on any quadric surface, for any distinct values z_1 , z_2 , z_3 :*

$K =$

$y \diagup x$	z_1	z_2	z_3
z_1	z_1	z_2	z_1
z_2	z_1	z_2	z_2
z_3	z_1	z_2	z_3

The reason for this is simple: it will be shown that the standard table of 3-algebra on a 2-surface obeys a "diagonal relation"

$$z_{12} + z_{23} + z_{31} = z_{21} + z_{32} + z_{13} = L$$

When applied to this algebra, one concludes that $z_1 = z_2$, which is prohibited in a faithful representation.

The coefficients a_{ij} of the m^{th} degree polynomial P(x,y) can be treated as free parameters. The general polynomial in two variables of the m^{th} degree has k=

(m+1)(m+2)/2 coefficients , as is easily verified by mathematical induction, (some of these may of course be zero). The standard table T of an n^{th} order composition **algebra will has n^2 entries. The representation of the algebra K^n on an m^{th} degree polynomial surface produces a system of n^2 equations in n + (m+1)(m+2)/2 variables:**

$$E_{\alpha\beta}: a_{m0} z_\alpha^m + a_{0m} z_\beta^m + a_{m-1,1} z_\alpha^{m-1} z_\beta +$$

$$a_{m-2,2} z_\alpha^{m-2} z_\beta^2 + \ldots + a_{0,1} z_\beta^1 + a_{00} (= q)$$

$$= z_{\alpha\beta} \varepsilon Z = \{z_1, z_2, \ldots\ldots, z_n\}$$

$$\alpha, \beta = 1, 2, 3, \ldots\ldots\ldots n$$

LEMMA:

If $R_K^n = \{a_{ij}; z_\gamma; 0 \le i + j \le n; \gamma = 1, 2, \ldots n)$ **is any polynomial representation of the nth order algebra K, then**

$\left\langle R_K^n \right\rangle_h = \{a_{ij}(1/h^{(i+j)}), ha_{00}, hz_\gamma; 0 < i + j \le n; \gamma = 1, 2, \ldots n)$ **, where h is any non-zero number, is also a representation. We might call this *property polynomial homogeneity* . Because of this Lemma, the number of independent variables in the system E is reduced to**

ν = (m+1)(m+2)/2 + n -1

Let us examine the table of values for the number of table entries ,n^2 , beside the number of coefficients (n+1)(n+2)/2, up to n = 12

TABLE I

n, m	n^2	$(m+1)(m+2)/2$
1	1	3
2	4	*6
*3	*#9	#10
4	16	15
!5	!25	!21
6	@36	28
7	49	@36
8	64	45
9	81	55
10	100	66
11	121	92
12	144	105

Interesting situations of have been given diacritical marks:

*** (i) 3-algebras on 2-surfaces. In this case, the number of table entries is 9, while the number of variables is 6 (coefficients) + 3(roots) is also 9. However, by the above lemma, the total number of free variables is only 8. Because of this fact only certain marginal 3-algebras can be represented on 2-surfaces. These algebras will be the subject of the next section .**

#(ii) 3-algebras on 3-surfaces. The #of table entries is 9, while the number of free coefficients is 10.

@(iii) 6-algebras on 7th degree surfaces. Here the number of table entries = the number of coefficients = 36. This situation is not repeated again until we come to algebras of order 204 (=12x17), and surfaces of degree 287 (! $m^2 = (12x17)^2 =$

(288x289)/2) . The general solution is equivalent to all solutions of equations of the form

$k^2 - 8m^2 = 1$, where $k = 2n+3$.

(iv) 5-algebras on 5^{th}-degree surfaces. This case is unique ,since : $n^2 = (n+1)(n+2)/2 + n - 1$; the number of independent variables is *exactly equal* to the number of equations . This is of great interest though it is probably very difficult to deal with . The indications suggest that , for the general fifth order algebra, the number of representations on 5th degree surfaces is given by a finite number of points in projective 4-space $P^4 = (p_1, p_2, p_3, p_4)$, where $p_j = z_j / z_5$.

Case (iii) is the simplest, while Case (ii) shares essentially the same features:

THEOREM XI :

If p= (z1,.....,z6) is a general point in complex C^6 space, let D^6 be that subspace consisting of all points in C^6 for which the determinant

$$\Delta = \left\| z_i^{\alpha} z_j^{\beta} \right\| \neq 0, 0 < \alpha + \beta \leq 7, i, j = 1,2,..6$$

$$= \begin{vmatrix} z_1^7 & z_1^7 & z_1^7 & \dots & \dots & z_1^1 & 1 \\ z_2^7 & z_2^7 & z_2^7 & \dots & \dots & z_2^1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ z_1^7 & z_2^7 & z_1^6 z_2^1 & \dots & \dots & z_2^1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ z_5^7 & z_6^7 & z_5^6 z_6^1 & \dots & \dots & z_6^1 & 1 \end{vmatrix}$$

does not equal zero. Then , if $p \in D^6$, and K^6 is any 6th order algebra, there exists a unique surface of 7th degree $z = S(x,y)$, that holds the algebra K^6 and is specified on the values $z_1,....z_6$.

The proof consists of a simple application of linear algebra. There is a corresponding theorem for the 3-algebras on 3rd degree surfaces:

Let $q = a_{00}$ be given , and let the point $p = \{ z_1, z_2, z_3 \}$ be so chosen that the determinant

$$\Delta = \left\| z_i^{\alpha} z_j^{\beta} \right\| \neq 0, 0 < \alpha + \beta \leq 3, i, j = 1,2,3$$

$$= \begin{vmatrix} z_1^3 & z_1^3 & z_1^3 & z_1^3 & z_1^2 & z_1^2 & z_1^2 & z_1 & z_1 \\ z_2^3 & \dots & \dots & & & & & & \\ z_3^3 & \dots & \dots & & & & & & \\ z_1^3 & z_2^3 & z_1^2 z_2 & z_1 z_1^2 & z_1^2 & z_2^2 & z_1 z_2 & z_1 & z_2 \\ z_1^3 & z_3^3 & \dots & \dots & & & & & \\ \dots & & & & & & & & \\ \dots & & & & & & & & \\ \dots & & & & & & & & \\ z_3^3 & z_2^3 & \dots & \dots & & & & z_3 & z_2 \end{vmatrix}$$

does not equal zero. Then, for any given 3-algebra K^3 there is a unique 3rd degree surface ,

$S(x,y) = z - q = ax^3+by^3+cx^2y+dxy^2+ex^2+fy^2+gxy+hx+iy$, holding K3 at the specified points (z_α , z_β , $z_{\alpha\beta} = S(z_\alpha , z_\beta)$)

$z\gamma \; \varepsilon \; p$, $\gamma = 1,2,3$ **. The result is once again based on the fact that the number of free coefficients in a 3rd degree polynomial of two variables, less its constant term, is equal to the number of tabular entries of a 3rd order algebra.**

It can be shown that this determinant is equal to :

$$\Delta = A z_1 z_2 z_3 (z_1 - z_2)^4 (z_2 - z_3)^4 (z_3 - z_1)^4 \bullet$$
$$(z_1^2 z_2 - z_1 z_2^2 + z_2^2 z_3 - z_3^2 z_2 + z_3^2 z_1 - z_1^2 z_3) \bullet$$
$$(2(z_1^2 + z_2^2 + z_3^2) + z_1 z_2 + z_1 z_3 + z_2 z_3)$$

where A is a constant.

It follows that if $p = (z_1, z_2, z_3)$ is not on the above collection of surfaces, then there is a 1-1 correspondence between 3-algebras specified on the values of *p* and 3-surfaces, z=f(x,y) passing through the set of composition points (x,y, f(x,y))

There are other simple relationships between order and degree to be discovered in the columns of Table I . An 8th order algebra has 64 tabular entries,

while an 11th order equation has 66 coefficients. This situation resembles that of the 2-algebras, with 4 entries, on 2-surfaces, with 6 coefficients.

A 10th order algebra has 100 entries, while a 13th order equation in 2 variables has 91 coefficients . Since 91+10 = 101, we have a situation similar to that of the 5th order algebras on 5th degree surfaces, wherein the number of really independent variables is equal to the number of table entries. Therefore $[(m+1)(m+2)/2]+n-1= n^2$, or $(m+1)(m+2) = 2(n^2 - n+1)$. We will solve the corresponding Diophantine equation in just a moment, but first we will find it useful enlarge Table I to the value N= 40, listing the quantities N or M , N^2 , N^2 - N+1 and (M+1)(M+2)/2 :

#68...

TABLE II

n, m	n^2	$n^2 - n + 1$	$(m+1)(m+2)/2$
1	1	1	3
2	4	3	6
3	9	7	10
4	16	13	15
5	25	21	21
6	36	31	28
7	49	43	36
8	64	57	45
9	81	73	55
10	100	91	66
11	12	111	78
12	144	133	91
13	169	157	105
14	196	183	120
15	225	211	136
16	256	241	153
17	289	273	171
18	324	307	190
19	361	343	210

20	400	381	.............231...............
21	441	421	253
22	484	463	276
23	529	507	300
24	576	553	325
25	625	601	351
26	676	651	378
27	729	703	406
28	784	757	435
29	841	813	465
30	900	871	496
31	961	931	528
32	1024	993	561
33	1089	1057	595
34	1156	1123	630
35	1225	1191	666
36	1296	1261	703
37	1369	.	741
38		.	780
39		.	820
40	1600	1561	861
.	.	.	.

Let Cm represent the number of coefficients in a bivariate polynomial of degree m .

Case 1 : $C_m = n^2$. The only solution of this is clearly m=7, n = 6. In this case one can identify a unique surface for a given algebra and specification of elements.

Case 2: $C_m = n^2 + 1$. Here one needs only specify the free constant q to obtain unique surfaces for given algebras and specified elements. The solutions of this are n=m=3 , and m=24, Cm = (25 · 26) /2 = 325 = $(18)^2$ + 1, so n= 18.

Case 3: $C_m = n^2 - n + 1$. This situation relates to a classical problem in number theory. We require:

(i) (m+1)(m+2) = 2(n^2 - n+1) , which simplifies to

(ii) m(m+3) = 2n(n-1). Let m = an+b, b<n, a ≥ 0, all integers. Substituting in (ii) gives

(iii) $(a^2 - 2)n^2 + (2ab + 3a + 2)n + b(b + 3) = 0$

For a=0 ,m must be less than n, and there is no solution. When a=1, there is only one solution, b=0, n=5,m=5. Proof: m<n ---> m+3 >2(n-1), or

n ≥ m ≥ 2(n-1)-3, or n> 2(n-1)-3. Continuing, we get n+3 > 2n-2, n ≤ 5, or n = 5.

Now a cannot be larger than 1; the coefficient (a2 -2) at the far lefthand guarantees that for a ≥ 2 the whole expression is positive, hence not equal to 0.

Setting a= 1 we can solve a simple quadratic equation that gives us n in terms of b. The solution is

$$n = (5 + 2b \pm \sqrt{(5 + 2b)^2 + 4b(b + 3)})/2$$

The term inside the square root is $8b^2 + 32b + 25 = g^2$. Define d by d = 2(b+2). The above expression then reduces to

$$2d^2 - g^2 = 7$$

The problem therefore reduces to adumbrating the even solutions to this classical equation. The first few solutions are:

d= 4 g = 5 n = 5 m = 5

d=8 g= 11 n = 10 m = 12

d=20 g = 19 n = 27 m = 36

....

We summarize these results in the following

THEOREM XII:

When (i) $2d^2 - g^2 = 7$

(ii) $b = d/2 - 2$

(iii) $n = (5+2b+\sqrt{8b^2 + 32b + 25}) / 2$

(iv) $m = n+b$

then the system of equations

$$E_{\alpha\beta}: a_{m0}z_\alpha^m + a_{0m}z_\beta^m + a_{m-1,1}z_\alpha^{m-1}z_\beta + a_{m-2,2}z_\alpha^{m-2}z_\beta^2 + \ldots + a_{0,1}z_\beta^1 + a_{00}(= q) = z_{\alpha\beta}\ \varepsilon\ Z = \{z_1, z_2, \ldots\ldots., z_n\};\ (\alpha, \beta = 1, 2, 3, \ldots\ldots\ldots n)$$

has n^2 free variables for n^2 equations. If we fix q, where q is the constant term, then any algebra K^n will lie on at most a finite number (perhaps none) of non-degenerate surfaces of mth degree in 3 space passing through the point (0, 0, q) .

3. Representations of Trivariate Algebras On Quadric Surfaces

We have singled out this case as one of exceptional interest to us for several reasons, among which:

(a) It is very simple. Many of the properties of these algebras and surfaces resemble those of the 2-algebras on quadratic surfaces

(b) In the situation of 1-dimensional rational maps, the quadratic case is the most interesting, being in some ways central to the dynamics of all rational maps.

(c) Since $C_m = 6$, $n = 3$ and $n^2 = 9$, that is to say,

 # Coefficients + # Elements = # Tabular Entries

, one almost , but not quite, obtains a 1-1 correspondance between algebras and surfaces. Write, as before:

$$(\alpha) az_1^2 + bz_1^2 + cz_1^2 + dz_1 + ez_1 + q = z_{11}$$

$$(\beta) az_2^2 + bz_2^2 + cz_2^2 + dz_2 + ez_2 + q = z_{22}$$

$$(\gamma) az_3^2 + bz_3^2 + cz_3^2 + dz_3 + ez_3 + q = z_{33}$$

$$(\delta) az_1^2 + bz_2^2 + cz_1z_2 + dz_1 + ez_2 + q = z_{12}$$

$$(\varepsilon) az_2^2 + bz_1^2 + cz_2z_1 + dz_2 + ez_1 + q = z_{21}$$

$$(\varsigma) az_1^2 + bz_3^2 + cz_1z_3 + dz_1 + ez_3 + q = z_{13}$$

$$(\eta) az_3^2 + bz_1^2 + cz_3z_1 + dz_3 + ez_1 + q = z_{31}$$

$$(\theta) az_2^2 + bz_3^2 + cz_2z_3 + dz_2 + ez_3 + q = z_{23}$$

$$(\mu) az_3^2 + bz_2^2 + cz_3z_2 + dz_3 + ez_2 + q = z_{32}$$

$$h = a + b + c$$

$$l = d + e$$

$$k_1 = z_2 + z_3$$

$$k_2 = z_1 + z_3$$

$$k_3 = z_1 + z_2$$

Once again the first 3 equations is the *outer set*, while the remaining 6 are the *inner set*. Given the values of the coefficients a, b, c, d, e, and q, there is enough information in the inner set to derive the three elements z_1, z_2, z_3, which may be called the "roots" of the algebra on the surface : those values at which the surface in some sense "intersects" the algebra.

Since the number of really independent variables of the system is only 8, these may be inconsistent with the values of the roots which can be derived by manipulating the inner equations. As it turns out, remarkably, the consistency of the roots as derived from the outer equations with the entire system, *is equivalent to a set of conditions tied up with the single coefficient, c. The c-conditions completely determine which 3- algebras can or cannot exist on a 2-surface.* After a fundamental

analysis of the system and consideration of several important special cases, the following theorem will have been proven by the end of the paper. Through it, the division of 3-algebras into those which can , and those which cannot exist on a 2-surface may be demonstrated :

Culminating Theorem :

A: This equation, called the "diagonal relation", must hold for any 3-algebra on a 2-surface:

$$z_{12} + z_{23} + z_{31} = z_{21} + z_{32} + z_{13} = L$$

B: If a 3-algebra K exists on a surface for which c = 0, then this set of equations must hold for all values of the indices α , β, γ , δ ε {1,2,3} :

$$z_{\alpha\beta} + z_{\gamma\delta} = z_{\gamma\beta} + z_{\alpha\delta}$$

C: If a 3-algebra K exists on a surface for which $c \neq 0$, then (A) must hold, but (B) cannot be satisfied by any (non-trivial) combination of the indices
α , β, γ , δ ε {1,2,3} .

D: If B holds for some sets of indices but not for others, or if A is not satisfied, or both, then the algebra K cannot be realized on any 2-surface.

Since condition D can be verified by a quick inspection, we have a method for deciding if any 3-algebra can be represented on a 2-surface. The class R of 3-algebras representable on 2-surfaces is a sub-class of the general class T of all 3-algebras which, as we have seen, are representable in the greatest generality by functions of 3 variables.

The Outer Equations

LEMMA I: If $h \neq 0$, there cannot be 3 identical trace entries, $z_{11} = z_{22} = z_{33}$.

PROOF: **A quadratic equation can't have 3 distinct roots**

LEMMA II: If $h \neq 0$, one cannot have $z_{11} = z_1$, $z_{22} = z_2$,

$z_{33} = z_3$.

PROOF: **Same as above. However in this case the quadratic equation in question is a different one. Exercise left to the conscientious reader!**

In the situation in which h = 0, one cannot use the outer equations to determine the values of the roots. The class of 3-algebras for which h does = 0 is interesting, and will be considered separately. For the moment, unless otherwise indicated, we will assume that $h \neq 0$.

LEMMA III:

(i) The *traces* of the standard tables of all 3-algebras on 2-surfaces for which $h \neq 0$ **are isomorphic to one of these 5:**

$$T_1 = (z_1, z_2, z_1)$$
$$T_2 = (z_2, z_1, z_1)$$
$$T_3 = (z_1, z_1, z_2)$$
$$T_4 = (z_1, z_3, z_2)$$
$$T_5 = (z_2, z_3, z_1)$$

(ii) There do exist algebras with traces (z_1, z_1, z_1) **and** (z_1, z_2, z_3) **on surfaces for which h = 0.**

PROOF : **If K has 2 fixed points then assume them to be** z_1 **and** z_2 **. The entry for** z_{33} **cannot then be** z_3 **and must therefore be either** z_1 **or** z_2 **. (The traces** $T_a = (z_1, z_2, z_1)$ **and** $T_b = (z_1, z_2, z_2)$ **are equivalent.)**

If K has one fixed point then the permissible traces are T_3 **or** T_4**, the trace** (z_1, z_1, z_1) **being ruled out. All other traces with a single fixed point are equivalent to these.**

Finally, all traces with no fixed points are isomorphic to either T_2 **or** T_5 **.**

(ii) Trivially, the plane surface whose equation is given by f(x,y) = z_1 **= Constant, will hold the constant algebra** $O^3_{z_1}$**, and will therefore have constant trace. Likewise, the plane surface whose equation is given by f(x,y) = x , will have trace** (z_1, z_2, z_3)**, where these 'roots' can be taken as any three distinct values.**

Since the trace entries are solutions of the outer equations, we may use these 5 trace forms to derive solutions for the 3 roots with respect to each one of them . In all cases but one, that of T_5, these turn out to be very simple expressions in the parameters a,b,c,d,e,q. The case of T_5 is *much* more complicated, but may be reduced to the solutions of a 6^{th} degree equation.

Expressions for the roots z1 , z2, z3 , in terms of the parameters a,...q , and traces T1,...T5 .

I.

$\underline{T_1 = (z_1, z_2, z_1):}$

$$(\alpha) hz_1^2 + lz_1 + q = z_1$$

$$(\beta) hz_2^2 + lz_2 + q = z_2$$

$$(\gamma) hz_3^2 + lz_3 + q = z_1$$

z_1, z_2 are the roots of the fixed point equations, and are given by :

$$z_1, z_2 = (1 - l \pm \sqrt{(1-l)^2 - 4qh}) / 2h$$
$$= (1 - l \pm g) / 2h$$

z_3 can be either of the two roots of the third equation, which depends on the two possible values of z_1, so that there are four possible expressions z_3. However, since two of these duplicate the values for z_1 and z_2 , we obtain two distinct solutions for z_3 which we may call z_3 , z_4:

$$z_3, z_4 = (-l \pm \sqrt{l^2 - 4h(q - z_1)})/2h$$
$$= (-l \pm \sqrt{l^2 - 4h(q - (1 - l \pm g)/2h)})/2h$$
$$= (-l \pm \sqrt{l^2 - 4hq + 2 - 2l + 2g})/2h$$
$$= (-l \pm \sqrt{(l-1)^2 + 1 \pm 2g - 4hq})/2h$$
$$= (-l \pm \sqrt{(l-1)^2 + 1 \pm 2\sqrt{(l-1)^2 - 4qh} - 4hq})/2h$$
$$= 1/2h(-l \pm (\sqrt{1 + (\sqrt{(l-1)^2 - 4qh})\ )^2}$$
$$= 1/2h(-l \pm \sqrt{(l-1)^2 - 4qh})$$
$$\therefore z_3, z_4 = (-1 - l \pm \sqrt{(l-1)^2 - 4qh})/2$$
$$= (-1 - l \pm g/2)$$

This result was also obtained in the first paper by other means.

II.

$\underline{T_2 = (z_2, z_1, z_1):}$

$$(\alpha) hz_1^2 + lz_1 + q = z_2$$

$$(\beta) hz_2^2 + lz_2 + q = z_1$$

$$(\gamma) hz_3^2 + lz_3 + q = z_1$$

Combining equations (α) and (β) with the results of Part I, we see that:

$$h(z_2 + z_3) + l = 0; z_2 + z_3 = -l/h$$

$$z_3 = -l/h - z_2 = (1 - l \pm \sqrt{(1-l)^2 - 4qh - 4})/2h$$

Since Equation (β) = Equation (γ) :

$$h(z_2 + z_3) + l = 0; z_2 + z_3 = -l/h$$

$$z_3 = -l/h - z_2 = (1 - l \pm \sqrt{(1-l)^2 - 4qh - 4})/2h$$

III.

$\underline{T_3 = (z_1, z_1, z_2):}$

$$(\alpha) hz_1^2 + lz_1 + q = z_1$$

$$(\beta) hz_2^2 + lz_2 + q = z_1$$

$$(\gamma) hz_3^2 + lz_3 + q = z_2$$

z_1 is the only fixed point. From Part I we have :

$$z_1 = (1 - l \pm \sqrt{(1-l)^2 - 4qh}) / 2h$$

$$z_2 = -(1 + l \pm \sqrt{(1-l)^2 - 4qh}) / 2h$$

$$z_1 + z_2 = k_3 = -l / h;$$

$$z_2 = -l / h - z_1, so$$

$$hz_3^2 + lz_3 + q = -l / h - z_1$$

$$\therefore z_3 = (1 - l \pm \sqrt{(l^2 - 4qh - 4l - 4hz_1)}) / 2h$$

$$= \ldots\ldots\ldots\ldots = \underline{\underline{1/2h(-l \pm \sqrt{(1-g)^2 - 4}}}$$

IV.

$\underline{T_4 = (z_1, z_3, z_2):}$

$$(\alpha) hz_1^2 + lz_1 + q = z_1$$

$$(\beta) hz_2^2 + lz_2 + q = z_3$$

$$(\gamma) hz_3^2 + lz_3 + q = z_2$$

z_1 is a fixed point, while z_2 , z_3 form a "toggle switch" set. Therefore:

$$z_1 = (1 - l \pm \sqrt{(1-l)^2 - 4qh}) / 2h$$

$$z_2, z_3 = (-1 - l \pm \sqrt{(1-l)^2 - 4qh - 4}) / 2h$$

V.

$\underline{T_5 = (z_2, z_3, z_1):}$

$$(\alpha) hz_1^2 + lz_1 + q = z_2$$

$$(\beta) hz_2^2 + lz_2 + q = z_3$$

$$(\gamma) hz_3^2 + lz_3 + q = z_1$$

Although this system of equations doesn't admit of simple solutions in terms of the parameters, there are some simplifying relations . Subtracting (β) from (α) gives:

$$(\alpha) - (\beta) = z_2 - z_3 = (z_1 - z_2)(h(z_1 + z_2) + l), or$$

$$h(z_1 + z_2) + l = (z_2 - z_3) / (z_1 - z_2)$$

Likewise, subtracting (g) from (b) and rearranging gives us:

$$h(z_2 + z_3) + l = (z_3 - z_1) / (z_2 - z_3)$$

$$= -1 - (z_1 - z_2) / (z_2 - z_3)$$

Therefore, $h(z_2 + z_3) + l = -1 - 1 / (h(z_1 + z_2) + l)$

Working through all the arithmetic we end up with this complicated expression:

$$z = \frac{-(h^2 z_2^2 + h^2 z_1 z_2 + h(l+1)z_1 + h(2l+1)z_2 + l^2 + l + 1)}{h(h(z_1 + z_2) + l)}$$

Plug this into the right side of (β), and substitute the left-hand side of (α) for the variable z_2 in (β). Our resulting equation will be of the 6th degree in z_1.

4. Restrictions On 3-Algebras Representable On Quadric Surfaces in 3-Space.

THEOREM XIII:

(a) If K^3 is a 3-algebra on a 2-surface, with standard table matrix M = { $z_{\alpha\beta}$ } , α,β = 1,2,3, then we must have:

$$z_{12} + z_{23} + z_{31} = z_{21} + z_{32} + z_{13} = L$$

This will be referred to as the "diagonal relation"

PROOF: **Referring to the tables of the outer and inner equations, we see that:**

$$(\delta) + (\theta) + (\eta) = z_{12} + z_{23} + z_{31} = az_1^2 + bz_2^2 + cz_1z_2 + dz_1 + ez_2 + q$$

$$+az_2^2 + bz_3^2 + cz_2z_3 + dz_2 + ez_3 + q$$

$$+az_3^2 + bz_1^2 + cz_3z_1 + dz_3 + ez_1 + q +$$

$$az_2^2 + bz_1^2 + cz_2z_1 + dz_2 + ez_1 + q = (\varepsilon) + (\varsigma) + (\mu) =$$

$$z_{21} + z_{32} + z_{13}$$

$Q.E.D.$

LEMMA: **If T is the trace and L the value of the diagonal relation, then :**

$$T - L = (z_{11} + z_{22} + z_{33}) - (z_{21} + z_{32} + z_{13}) =$$

$$c((z_1^2 + z_2^2 + z_3^2) - (z_1z_2 + z_1z_3 + z_2z_3))$$

PROOF: **Add the outer equations and subtract from the sum one of the two combinations that form the diagonal sum L.**

THEOREM XIV :

(a) If c= 0, then T=L

(b) If z_1 ,z_2 ,z_3 are all real numbers , then

$$c > 0 \rightarrow T > L \quad \& \quad c < 0 \rightarrow T < L$$

(a) follows immediately from the lemma; (b) follows from the fact that the right hand expression in the lemma is positive definite. Note that the theorem

is no longer true if we allow the roots to be complex. We have thus uncovered a distinction between 'real' and 'complex' dynamics of 3-algebras on a 2-surface.

This theorem is useful for determining whether or not a given 3-algebra with real roots can live on a real 2-surface. For example, if c>0, z_1 is the smallest of the 3 real roots ,and the table has traces T_1, T_2, or T_3, then we must also have $z_1 = z_{12} = z_{21} = z_{13} = z_{31}$ **It is not difficult to show that this is incompatible with both $c \neq 0$, and also c=0. The only possibility then is that this is the constant algebra O_{z1}, whose trace is**

$T = (z_1, z_1, z_1)$.

5. Expressing the Coefficients in Terms of the Roots

When the set of 9 equations is solved for the 6 coefficients in terms of the 3 roots, one derives quite simple expressions for a,b,d,e,q. The evaluation of c however gives us a long chain of equalities which must be adjusted to each other. This sets up relations between the roots which yield definitive criteria for deciding whether or not any given 3-algebra K can be represented on some 2-surface at all. Obviously the diagonal relation is one such criterion. Another criterion is, obviously: ***if some of the c-relations imply that c=0, while others imply that $c \neq 0$, then the algebra in question clearly cannot be represented on any 2-surface.***

We shall show that these criteria are both necessary and sufficient.

Definition : **Consider the the diagonal relation:**

$$z_{12} + z_{23} + z_{31} = z_{21} + z_{32} + z_{13} = L$$

One chooses values for these double-index terms within the set $Z^3 = (z_1, z_2, z_3)$. If the left hand side of the relation is identical to the right side when choices are treated as indeterminates, then we will say that the two sides are ***formally equal*** **. However, suppose we let**

$$z_{12} = z_{23} = z_{31} = z_1$$

$$z_{21} = z_1; z_{32} = z_2; z_{13} = z_3$$

This choice of the diagonal terms will be compatible with the diagonal relation if we restrict the roots by the equation :

(I): $z_1 = (z_2 + z_3)/2$ **. In this case , z1 will be called the *averaging element* .**

We may also select a combination of entries of the form:

$$z_{12} = z_{23} = z_{31} = z_1$$

$$z_{21} = z_2; z_{32} = z_2; z_{13} = z_3$$

Then the roots will be related by the equation

(II): $z_1 = (2z_2 + z_3)/3$

In these two cases, which are , save for permutations on the indices, the only ones, we will say that the two sides are *algebraically identical* , and say that the diagonal relation is *algebraically satisfied* .

THEOREM XV : (I) and (II) and all of their the variants obtained by permutations of the indices are mutually exclusive. Since there are 3 forms of (I) and 6 forms of (II) this means that there are 9 mutually exclusive possibilities for an algebraic solution of the diagonal relation.

PROOF: **It is clearly that the variants of (I) must be mutually exclusive (It being always assumed that the roots are distinct), since if one among 3 distinct quantities is the average of the 2 others, neither of the other two can also be the average of the remaining ones . A similar argument shows that the 6 variants of 2 must also be exclusive. We need therefore only examine the following cases**

$$(i) \quad z_1 = {(z_2 + z_3)}/{2} \;\&\; z_1 = {(2z_2 + z_3)}/{3}$$

$$(ii) \; z_1 = {(z_2 + z_3)}/{2} \;\&\; z_2 = {(2z_1 + z_3)}/{3}$$

$$(iii) z_1 = {(z_2 + z_3)}/{2} \;\&\; z_2 = {(z_1 + 2z_3)}/{3}$$

All three cases lead to the conclusion that $z_1 = z_2 = z_3$, and we may consider the theorem proven.

The c-Relations

THEOREM XVI : **If K is any 3-algebra on a 2-surface with equation given by**

$$z = f(x,y) = ax^2 + by^2 + cxy + dx + ey + q$$

,then

$$c = (z_{\alpha\beta} + z_{\gamma\delta} - z_{\gamma\beta} - z_{\alpha\delta}) / (z_\alpha - z_\gamma)(z_\beta - z_\delta)$$
$$\alpha, \beta, \gamma, \delta = 1,2,3; \alpha \neq \gamma \ \& \ \beta \neq \delta$$

PROOF: The proof follows immediately from the addition and subtraction of the equations corresponding to the double-index terms:

$$z_{\alpha\beta} + z_{\gamma\delta} - z_{\gamma\beta} - z_{\alpha\delta} =$$
$$(az_\alpha^2 + bz_\beta^2 + cz_\alpha z_\beta + dz_\alpha + ez_\beta + q) +$$
$$(az_\gamma^2 + bz_\delta^2 + cz_\gamma z_\delta + dz_\gamma + ez_\delta + q) -$$
$$(az_\alpha^2 + bz_\delta^2 + cz_\alpha z_\delta + dz_\alpha + ez_\delta + q) -$$
$$(az_\gamma^2 + bz_\beta^2 + cz_\gamma z_\beta + dz_\gamma + ez_\beta + q)$$
$$= c(z_\alpha z_\beta + z_\gamma z_\delta - z_\alpha z_\delta - z_\gamma z_\beta)$$
$$= c(z_\alpha - z_\gamma)(z_\beta - z_\delta)$$

Very few of these equations are independent. We will show later that they reduce to only 3 independent equations,
(including the diagonal relation). For the moment we will use these relations to completely characterize 2 classes of algebras of special interest:

(1) Those for which c=0, h $\neq$ 0

(2) Those for which c = h = 0

Theorem XVII : **c= 0 for all 1-dimensional algebras**

Proof: If K^3 is a 1-dimensional 3-algebra, then f(x,y) = g(x), or f(x,y) = g(y) for all combinations of the 3 roots. Assuming the former case (right-insensitive), then:

$$z_{11} = f(z_1, z_1) = g(z_1) = f(z_1, z_2) = z_{12}$$

$$z_{21} = f(z_2, z_1) = g(z_2) = f(z_2, z_2) = z_{22}$$

Therefore

$$z_{11} + z_{22} - z_{12} - z_{21} = 0$$

This is the c-relation for c=0 Q.E.D.

Lemma : If c=0 and all of the terms in any row, or in any column, are identical, then K is a 1-dimensional algebra.

Proof: With no loss of generality, let us say that $z_{11}=z_{12}=z_{13}=z_{\alpha}$. Then:

$$az_1^2 + bz_1^2 + dz_1 + ez_1 + q$$

$$= az_1^2 + bz_2^2 + dz_1 + ez_2 + q$$

$$= az_1^2 + bz_3^2 + dz_1 + ez_3 + q; so,$$

$$b(z_1 + z_2) + e = 0$$

$$b(z_2 + z_3) + e = 0$$

$$\therefore b(z_1 - z_3) = 0, and, because$$

$$z_1 \neq z_3,$$

$$\therefore b = 0, and, e = 0$$

This proof can be generalized to all other cases.

Theorem XVIII: If c=0, h $\neq$ 0 , and K is 2-dimensional, then K must be of the form:

$$K = \begin{array}{c|c|c|c} y/x & z_1 & z_2 & z_3 \\ \hline z_1 & z_\gamma & z_\beta & z_\beta \\ \hline z_2 & z_\beta & z_\alpha & z_\alpha \\ \hline z_3 & z_\beta & z_\alpha & z_\alpha \end{array}$$

with

$\alpha, \beta, \gamma \in (1,2,3), and$

$z_\beta = (z_\alpha + z_\gamma)/2$

PROOF: This is the first long proof in this paper! I'm sure it can be simplified. Here is the long,lazy proof, with every intention, time willing, of reducing it to a few lines and a footnote!

Write out the c-relations for c= 0

$$z_{11} + z_{22} = z_{21} + z_{12}$$

$$z_{11} + z_{33} = z_{31} + z_{13}$$

$$z_{33} + z_{22} = z_{23} + z_{32}$$

$$z_{11} + z_{23} = z_{13} + z_{21}$$

$$z_{12} + z_{23} + z_{31} = z_{21} + z_{32} + z_{13}$$

It's clear that all of the other relations can be derived from these. If h $\neq$ 0, then the situations $z_{11}=z_1$, $z_{22}=z_2$, $z_{33}=z_3$ and $z_{11}=z_{22}=z_{33}=z\gamma$ (γ = *1,2,or 3*) are excluded.On can therefore write with no loss of generality:

$$z_{11} = z_\gamma \,\&\, z_{22} = z_\alpha ; \alpha \neq \gamma .$$

Therefore,

$$z_{12} + z_{21} = z_\alpha + z_\gamma$$

CASE I:

$$z_{12} = z_\alpha \& z_{21} = z\gamma (or, z_{12} = z_\gamma \& z_{21} = z_\alpha)$$

This is the situation of formal equality in the first c-relation. If K isn't a 1-dimensional algebra, then by the lemma, we can't have $z_{31} = z\gamma$, or $z_{32} = z\alpha$. However, we also have the c-relation: $z_{11} + z_{32}=z_{12} + z_{31}$. By substitution we derive $z\gamma + z_{32}=z\alpha + z_{31}$. Since we know that $z_{32} \neq z\alpha$ and $z31 \neq z\gamma$, the remaining possibilities are:

(a) $z_{32} = z\gamma$

(b) $z_{32} = z_\beta$

By the symmetries inherent in our construction, if we show that (a) is inadmissable and $z\alpha$ cannot be an averaging element, then we have also shown that (b) is inadmissable, so that $z\gamma$ also cannot be an averaging element. Indeed we have shown more: *no trace entry can be the averaging element*. Evidently the initial assumption that the first c-relation was a formal identity is wrong, and we must therefore assume $z_{12} = z_{21} = z\beta$, which then serves as the averaging element. We therefore examine the consequences of (a) :

$$\begin{aligned}
&If\ z_{32} = z_\gamma\ , then: \\
&2z_\gamma = z_\alpha + z_{31} \\
&\therefore z_{31} \neq z_\alpha, z_{31} \neq z_\gamma \rightarrow \underline{z_{31} = z_\beta}, and \\
&\underline{\underline{z_\gamma = (z_\alpha + z_\beta)/2.}} \\
&Therefore: \\
&(i) z_{13} + z_{31} = z_{11} + z_{33} \rightarrow z_{13} + z_\beta = z_\gamma + z_{33} \\
&(ii) z_{23} + z_{32} = z_{22} + z_{33} \rightarrow z_{23} + z_\gamma = z_\alpha + z_{33}
\end{aligned}$$

The table of K now looks like this:

$y \diagup x$	z_1	z_2	z_3
z_1	z_γ	z_α	$z_?$
z_2	z_γ	z_α	$z_?$
z_3	z_β	z_γ	$z_?$

We now make use of the diagonal relation to give us the remaining right-most column entries:

$$z_{11} + z_{22} + z_{33} = z_\alpha + z_\gamma + z_{33} = z_{12} + z_{23} + z_{31} =$$

$$z_\alpha + z_\beta + z_{23} = z_{21} + z_{32} + z_{13} = z_\alpha + z_\gamma + z_{13}$$

$$\therefore z_{13} = z_{33}$$

$$z_\beta + z_{23} = z_\gamma + z_{13} = z_\gamma + z_{33}$$

One cannot allow the possibility $z_{13} = z_{23}$ a ; by the Lemma, this gives a 1-dimensional algebra. This leaves only two possibilities:

$$(i) z_{33} = z_\beta ; z_{23} = z_\gamma$$

$$(ii) z_{33} = z_\gamma ; z_{23} = z_\alpha$$

$A^1 =$

y/x	z_1	z_2	z_3
z_1	z_γ	z_α	z_β
z_2	z_γ	z_α	z_γ
z_3	z_β	z_γ	z_β

$A^2 =$

y/x	z_1	z_2	z_3
z_1	z_γ	z_α	z_γ
z_2	z_γ	z_α	z_α
z_3	z_β	z_γ	z_γ

Plugging these into the diagonal relation we find that, in the first instance:

$$z_\alpha + z_\beta + z_\gamma = 2z_\gamma + z_\beta \rightarrow z_\alpha = z_\gamma \ ,$$

and in the second :

$$Trace = z_\alpha + z_\gamma + z_\gamma = L = z_{12} + z_{23} + z_{31}$$

$$= 2z_\alpha + z_\gamma \rightarrow z_\alpha = z_\gamma$$

This is a contradiction, since the three roots must be distinct. We have thus shown that

(i) A 2-dimensional algebra with c=0, $h \neq 0$, must have an averaging element;

(ii) This element cannot be on the trace, and therefore

(iii) z_β must be our averaging element = $(z_\alpha+z_\gamma)/2$

An examination of the c-relations for c=0, combined with this information, very quickly shows that *there is only one form that such a (2-dimensional) algebra can have*:

$K =$

y/x	z_1	z_2	z_3
z_1	z_γ	z_β	z_β
z_2	z_β	z_α	z_α
z_3	z_β	z_α	z_α

Q.E.D !!

6. The 2nd Degree Polynomials of 2-dimensional 3- algebras for which c=0,h ≠ 0

By the previous theorem, the table matrices of all these algebras have the same format. Combined with different vertical and horizontal borders, these provide 6 distinct tables, isomorphic in pairs, hence 3 really different algebras:

$A^1 =$

$y \diagup x$	z_1	z_2	z_3
z_1	z_1	z_2	z_2
z_2	z_2	z_3	z_3
z_3	z_2	z_3	z_3

; $A^2 =$

$y \diagup x$	z_1	z_2	z_3
z_1	z_1	z_3	z_3
z_2	z_3	z_2	z_2
z_3	z_3	z_2	z_2

$A^3 =$

$y \diagup x$	z_1	z_2	z_3
z_1	z_2	z_3	z_3
z_2	z_3	z_1	z_1
z_3	z_3	z_1	z_1

; $A^4 =$

$y \diagup x$	z_1	z_2	z_3
z_1	z_3	z_2	z_2
z_2	z_2	z_1	z_1
z_3	z_2	z_1	z_1

$A^5 =$

$y \diagup x$	z_1	z_2	z_3
z_1	z_3	z_1	z_1
z_2	z_1	z_2	z_2
z_3	z_1	z_2	z_2

; $A^6 =$

$y \diagup x$	z_1	z_2	z_3
z_1	z_2	z_1	z_1
z_2	z_1	z_3	z_3
z_3	z_1	z_3	z_3

The following result is presented without proof, as it can be derived very simply from the methods developed in this paper and in Part I:

THEOREM XIX :

(a) : The algebra A^1 is represented by the family of surfaces given by

$$z = f(x,y) =$$

$$a(x^2 + y^2) + d(x + y) + (4d^2 - 4d - 3)/8a$$

with

$$z_1 = (3 - 2d)/4a; z_2 = -(1 + 2d)/4a; z_3 = (1 - 2d)/4a$$

(b) : The algebra A^3 is represented by the family of surfaces given by:

$$z = f(x,y) =$$
$$a(x^2+y^2)+d(x+y)+(4d^2-4d-7)/8a$$
$$with$$
$$z_1 = -(3+2d)/4a; z_2 = (1-2d)/4a; z_3 = -(1+2d)/4a$$

(c) : The algebra A^5 is represented by the family of surfaces given by:

$$z = f(x,y) =$$
$$a(x^2+y^2)+d(x+y)+(d^2-d-2)/2a$$
$$with$$
$$z_1 = -d/2a; z_2 = (2-d)/2a; z_3 = -(2+d)/2a$$

Observe that algebras A^1, A^2, A^3 and A^4 a*re specified on the same set of roots z_1, z_2, z_3 for given a and d.*

7. The Universal 4-Algebra for c=0

In each of the above situations, there turns out to be a superfluous root, z_4. In fact, everyone of these 3-algebras is embedded as a sub-algebras of a more general 4-algebra, whose full table is given as follows:

y/x	z	z_1	z_2	z_3
z_1	z_γ	z_β	z_β	z_γ
z_2	z_β	z_α	z_α	z_β
z_3	z_β	z_α	z_α	z_β
z_4	z_γ	z_β	z_β	z_γ

$$z_\beta = (z_\alpha + z_\gamma)/2; \alpha,\beta,\gamma \in (1,2,3)$$
$$A^1: z_4 = -(3+2d)/4a$$
$$A^3: z_4 = (3-2d)/4a$$
$$A^5: z_4 = z_1$$

In the case of A^5 , the 4-algebra reduces to the basic 3-algebra , since $z_3 = z_4$ is the double root of a quadratic equation.

#90...

CASE 2:

c= h = 0

The primary feature of the degenerate case h=0, is that the outer equations collapse. The 3 linear equations that replace them limit the traces to a small set. Since a+b+c=0, we can write down the outer equations as:

$$z_{11} = lz_1 + q$$
$$z_{22} = lz_2 + q$$
$$z_{33} = lz_3 + q$$

If l ≠ 0 , then all of the trace elements are distinct. If l =0 then all of the trace elements are identical. Therefore, when h = 0, one cannot have a trace with entries drawn from 2 and only 2 members of the set Z^3 . This limits the possibilities effectively to four:

$$I\ldots z_{11} = z_{22} = z_{33} = z_1$$
$$II..z_{11} = z_1; z_{22} = z_2; z_{33} = z_3$$
$$III.z_{11} = z_1; z_{22} = z_3; z_{33} = z_2$$
$$IV..z_{11} = z_2; z_{22} = z_3; z_{33} = z_1$$

It is a simple matter to show that possibility IV will not work:

$$z_2 = lz_1 + q = l(lz_3 + q) + q =$$
$$l(l(lz_2 + q) + q) + q = l^3 z_2 + l^2 q + lq + q$$
$$= l^3 z_2 + q(l^3 - 1)/(l - 1)$$
$$\therefore z_2 = q/(1 - l) = z_1 = z_3, or$$
$$l^3 = 1$$

Unless l = 1, the solution $z_2 = q/(1 - l) = z_1 = z_3$ **must be the only solution, the equations being linear. If l = 1, then we have the requirement that:**

$$z_2 = z_1 + q = z_3 + 2q = z_2 + 3q =$$
$$z_1 + 4q = z_3 + 5q = z_2 + 6q = \ldots(!), or$$
$$q = 0, and$$
$$z_2 = z_1 = z_3$$

Possibility III can be eliminated by the same argument used to show that the averaging element of an algebra for which c=0 (and the diagonal r elation depends upon the averaging relation) cannot be on the trace.

What remains are the two exceptional traces, namely
I = (z_1 ,z_1 ,z_1) and II = (z_1, z_2, z_3)

Trace I: Invoking the set of c-relations for c=0 , one sees that:

$$z_{12} + z_{21} = z_{13} + z_{31} = z_{23} + z_{32} = 2z_1$$
$$z_{12} + z_{23} + z_{31} = z_{21} + z_{32} + z_{13} = 3z_1$$

These equations imply the following alternatives, up to isomorphism:

(1) K is the constant algebra, A^1:

$$z_{ij} = z_1, \forall i, j.$$

(2) K is the algebra given by

$$z_{12} = z_2; z_{21} = z_{3;} z_1 = (z_2 + z_3)/2$$

Once again, (2) admits of only two really distinct possibilities, which we may call A^2 and A^3 :

#92...

$A^2 =$

$y \backslash x$	z_1	z_2	z_3
z_1	z_1	z_1	z_2
z_2	z_1	z_1	z_2
z_3	z_3	z_3	z_1

$A^3 =$

$y \backslash x$	z_1	z_2	z_3
z_1	z_1	z_3	z_3
z_2	z_2	z_1	z_1
z_3	z_2	z_1	z_1

Although these two algebras resemble each other, they are not isomorphic. We can calculate the coefficients of the equations for A^2 and A^3 from the inner equations.

For A^2 we get:

$$z_1 = (z_2 + z_3)/2$$
$$a(z_1^2 - z_2^2) + d(z_1 - z_2) + z_1 = z_{12} = z_1$$
$$a(z_1^2 - z_3^2) + d(z_1 - z_3) + z_1 = z_{13} = z_2$$
$$\therefore a(z_1 + z_2) + d = (z_1 - z_1)/(z_1 - z_2) = 0$$
$$a(z_1 + z_3) + d = (z_2 - z_1)/(z_1 - z_3) = 1$$
$$\therefore a = 1/(z_3 - z_2), d = (z_1 + z_2)/(z_2 - z_3)$$
$$z = f(x,y) =$$
$$\frac{(x + y - z_1 - z_2)(x - y)}{z_3 - z_2} + z_1$$

For A^3 we get:

$$z_1 = (z_2 + z_3)/2$$

$$a(z_1^2 - z_2^2) + d(z_1 - z_2) + z_1 = z_{12} = z_3$$

$$a(z_1^2 - z_3^2) + d(z_1 - z_3) + z_1 = z_{13} = z_3$$

$$\therefore a(z_1 + z_2) + d = (z_3 - z_1)/(z_1 - z_2) = 1$$

$$a(z_1 + z_3) + d = (z_3 - z_1)/(z_1 - z_3) = -1$$

$$\therefore a = 2/(z_2 - z_3), d = 1 - 2(z_1 + z_2)/(z_2 - z_3)$$

$$= (z_2 - z_3 - 2z_1 - 2z_2)/(z_2 - z_3) =$$

$$-3(z_2 + z_3)/(z_2 - z_3)$$

Finally we look at Trace II , $z_{ii} = z_i$, i = 1,2,3. All 3 roots are fixed points. Examining the combinatorial possibilities in the manner already described we find that the only algebra of this sort which is not 1-dimensional, and for which c=h=0, is given by:

$K =$

$y \diagup x$	z_1	z_2	z_3
z_1	z_1	z_1	z_3
z_2	z_2	z_2	z_1
z_3	z_1	z_1	z_3

PROOF: **the c-relations give:**

$$z_{11} + z_{22} = z_1 + z_2 = z_{12} + z_{21}$$

$$z_{11} + z_{33} = z_1 + z_3 = z_{13} + z_{31}$$

$$z_{22} + z_{33} = z_2 + z_3 = z_{23} + z_{32}$$

$$z_{11} + z_{22} + z_{33} = z_1 + z_2 + z_3 =$$

$$z_{12} + z_{23} + z_{31} = z_{21} + z_{32} + z_{13}$$

Assume first that the equalities are formal, so that no root is required to be an averaging element. To avoid 1-dimensionality, one must have:

$z_{12} = z_1, z_{21} = z_2, z_{13}z_3, z_{31} = z_1,$ **, or its isomorphic equivalent.**

Combining this with the diagonal relation we get:

$$2z_1 + z_{23} = z_2 + z_3 + z_{32}$$

This in turn implies only two possibilities:

(1) z_1 is an averaging element = (z_2+z_3)/2

(2) z_{23} = z_3 and z_{32} = z_2

The situation produced by possibility (2) breaks down since $z_{13} = z_{23} = z_{33} = z_3$, which, for c=0 implies that the algebra is 1-dimensional, leading to the prohibited equality $z_1 = z_2$.

The above standard table , (and its isomorphisms), therefore describes the only possible 2-dimensional algebra for which c=h=0, and with trace T = (z_1 , z_2 , z_3) . Its equation is readily calculated:

$$z = f(x,y) = a(x^2 - y^2) + dx + ey + q$$

$$(d+e-1)z_1 + q = (d+e-1)z_2 + q = (d+e-1)z_3 + q = 0$$

$$\therefore d+e-1 = q = 0$$

$$z_1 = a(z_1^2 - z_2^2) + dz_1 + (1-d)z_2, or$$

$$a(z_1 + z_2) + (d-1) = 0$$

$$a(z_1^2 - z_3^2) + dz_1 + (1-d)z_3 = z_3, or$$

$$a(z_1 + z_3) + d = 0, and$$

$$z_1 = (z_2 + z_3)/2$$

$$\therefore a(z_2 - z_3) = 1, a = 1/(z_2 - z_3)$$

$$d = 1 - a(z_1 + z_2) = 1 - (z_1 + z_2)/(z_2 - z_3)$$

$$= (z_2 - z_3 - ((z_2 + z_3)/2 + z_2)/(z_2 - z_3)$$

$$= -(z_2 + 3z_3)/(z_2 - z_3)$$

The structure of this algebra is interesting. It contains two 2nd order subalgebras, both 1-dimensional, although the full 3-algebra is 2-dimensional

$A_1^2 =$

y/x	z_1	z_2
z_1	z_1	z_1
z_2	z_2	z_2

$A_2^2 =$

y/x	z_1	z_3
z_1	z_1	z_3
z_3	z_1	z_3

8. Complete Solutions for the Coefficients a, b, c, e, f, q , in Terms of the Roots z_1 , z_2 , z_3

We are assuming that the 3- algebra K does exist on some 2nd degree surface. This being the case, there will be no inconsistencies in the c-relations, and the other coefficients may be computed using simple linear algebra. Let

h= a+b+c ; l = d + e , q = constant term. Then the outer equations become :

$$(\alpha) hz_1^2 + lz_1 + q = z_{11}$$

$$(\beta) hz_2^2 + lz_2 + q = z_{22}$$

$$(\gamma) hz_3^2 + lz_3 + q = z_{33}$$

Treating these as a system of linear equations in the unknowns h,l, and q, we find:

$$\Delta = \begin{vmatrix} z_1^2 & z_1 & 1 \\ z_2^2 & z_2 & 1 \\ z_3^2 & z_3 & 1 \end{vmatrix} = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$$

$$h = \begin{vmatrix} z_{11} & z_1 & 1 \\ z_{22} & z_2 & 1 \\ z_{33} & z_3 & 1 \end{vmatrix} / \Delta$$

$$= \{z_{11}(z_2 - z_3) - z_{22}(z_1 - z_3) + z_{33}(z_1 - z_2)\} / \Delta$$

$$l = -\begin{vmatrix} z_{11} & z_1^2 & 1 \\ z_{22} & z_2^2 & 1 \\ z_{33} & z_3^2 & 1 \end{vmatrix} / \Delta$$

$$= \{z_{11}(z_3^2 - z_2^2) - z_{22}(z_3^2 - z_1^2) + z_{33}(z_2^2 - z_1^2)\} / \Delta$$

$$q = \begin{vmatrix} z_{11} & z_1 & z_1^2 \\ z_{22} & z_2 & z_2^2 \\ z_{33} & z_3 & z_3^2 \end{vmatrix} / \Delta$$

$$= \{z_{11}(z_2^2 z_3 - z_3^2 z_2) - z_{11}(z_1^2 z_3 - z_3^2 z_1) + z_{11}(z_1^2 z_2 - z_2^2 z_1)\} / \Delta$$

We may combine this with the inner equations to find a,b,d,e. Write h = h(z), l=l(z), q=q(z), where h is of degree -1, l of degree 0, and q of degree 1 in the roots z_1 z_2 z_3 . Forming the differences of the transpose entries, we have:

$$z_{12} - z_{21} = (a - b)(z_1^2 - z_2^2) + (d - e)(z_1 - z_2), or$$

$$(a - b)(z_1 + z_2) + (d - e) = (z_{12} - z_{21}) / (z_1 - z_2) = \lambda_1$$

$$(a - b)(z_2 + z_3) + (d - e) = (z_{23} - z_{32}) / (z_2 - z_3) = \lambda_2$$

These can be solved as ordinary linear equations in the unknowns a-b and d-e :

$$a - b = (\lambda_1 - \lambda_2)/(z_1 - z_3)$$

$$d - e = ((z_1 + z_2)\lambda_2 - (z_2 + z_3)\lambda_1)/(z_1 - z_3)$$

Combining this with the expressions

a+b = h(z) - c

d+e = l(z)

we can write the complete expressions for the coefficients as:

$$a = 1/2[(\lambda_1 - \lambda_2)/(z_1 - z_3) + h(z) - c]$$
$$b = 1/2[(\lambda_1 - \lambda_2)/(z_1 - z_3) - h(z) + c]$$
$$d = 1/2[((z_1 + z_2)\lambda_2 - (z_2 + z_3)\lambda_1)/(z_1 - z_3) + l(z)]$$
$$e = 1/2[((z_1 + z_2)\lambda_2 - (z_2 + z_3)\lambda_1)/(z_1 - z_3) - l(z)]$$

❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋❋

Finally we discuss the problem of describing those situations in which the c-relations are consistent or inconsistent. Having done this, we should then be able to look at a 3-algebra K and determine if it can exist on some 2-surface, and also write down the coefficients of the equations which contain such algebras in terms of the roots.

We therefore conclude this paper with an analysis of the c-relations.

Let $\tau = (z_1 - z_2)/(z_2 - z_3)$

THEOREM:

$$L = z_{12} + z_{23} + z_{31} = z_{21} + z_{32} + z_{13} =$$

$$z_{33}(2+\tau)\tau/(1+\tau) + z_{22}(\tau^2 - 1)/\tau - z_{11}(1+2\tau)/(\tau+1)\tau$$

This equation expresses the diagonal relation as a linear sum of the trace elements with coefficients in τ **. To prove this, we write the c-relations in terms of** τ **:**

$$c(z_1^2 - z_2^2) = z_{11} + z_{22} - z_{12} - z_{21}$$
$$= ((z_1 - z_2)^2 / (z_1 - z_3)^2)(z_{11} + z_{33} - z_{13} - z_{31}) =$$
$$((z_1 - z_2)^2 / (z_2 - z_3)^2)(z_{22} + z_{33} - z_{23} - z_{32}) =$$
$$((z_1 - z_2) / (z_1 - z_3))(z_{11} + z_{23} - z_{13} - z_{21})$$
$$= etc\ldots\ldots Since$$
$$(z_1 - z_2) / (z_1 - z_3) = ((z_1 - z_2) / (z_2 - z_3))((z_2 - z_3) / (z_1 - z_3)) =$$
$$\tau / (1 + \tau)$$
$$\therefore z_{11} + z_{22} - z_{12} - z_{21}$$
$$= (\tau / (1 + \tau))^2 (z_{11} + z_{33} - z_{13} - z_{31}) =$$
$$\tau^2 (z_{22} + z_{33} - z_{23} - z_{32}) =$$
$$(\tau / (1 + \tau))(z_{11} + z_{23} - z_{13} - z_{21}) = etc\ldots\ldots$$

Combining this with the diagonal relation we have:

$$z_{11}+z_{22}-z_{12}-z_{21}$$
$$=(\tau/(1+\tau))^2(z_{11}+z_{33}+z_{12}+z_{21}+z_{23}+z_{32}-2L)=$$
$$\tau^2(z_{22}+z_{33}-z_{23}-z_{32})=$$
$$(\tau/(1+\tau))(z_{11}+z_{23}+z_{32}-L)=etc......Let$$
$$u=z_{12}+z_{21},$$
$$v=z_{23}+z_{32}.Then:$$
$$z_{11}+z_{22}-u$$
$$=(\tau/(1+\tau))^2(z_{11}+z_{33}+u+v-2L)=$$
$$\tau^2(z_{22}+z_{33}-v)=$$
$$(\tau/(1+\tau))(z_{11}+v-L)$$

The elimination u and v from the above equations gives the desired result. To complete the proof of the "Culminating Theorem" announced on page 15 , we express τ in terms of terms $\alpha = z_1/z_2$, and $\beta = z_3/z_2$. This is:

$$\tau=(z_1-z_2)/(z_2-z_3)=(\alpha-1)/(1-\beta)$$
$$\tau/(\tau+1)=(\alpha-1)/(\alpha-\beta)$$

Substituting into the equations used in the proof of the above theorem, (the details are left to the reader), we end up with this pair of equations for the solution of α and β in terms of the table entries:

$$Let:\quad u=z_{12}+z_{21};v=z_{23}+z_{32};w=z_{13}+z_{31}.Then$$
$$2L=u+v+w\quad Then:$$
$$[1]:\alpha(u+z_{33}-L)+\beta(v+z_{11}-L)=z_{11}+z_{33}-w$$
$$[2]:\alpha(w+z_{22}-L)+\beta(u-z_{11}-z_{22})=L-v-z_{11}$$

This set of equations will always have acceptable solutions except when:

$$\alpha = \beta; \alpha = 1; \beta = 1; or \Delta = 0, where$$
$$\Delta = (u + z_{33} - L)(u - z_{11} - z_{22}) - (v + z_{11} - L)(w + z_{22} - L)$$

The restrictions on α and β are precisely the requirement that the 3 roots be distinct. Let α = 1. These equations reduce to:

$$[1]: u + z_{33} - L) + \beta(v + z_{11} - L) = z_{11} + z_{33} - w$$
$$[2]: w + z_{22} - L + \beta(u - z_{11} - z_{22}) = L - v - z_{11}$$
$$\beta(v + z_{11} - L) = z_{11} - u - w + L = z_{11} + v - L$$
$$\beta(u - z_{11} - z_{22}) = 2L - w - v - z_{11} = u - z_{11} - z_{22}$$

These equations are compatible only with either β = 1 or $L = v+z_{11}$, and $u=z_{11}+z_{22}$. The latter equation is one of the c-conditions for c=0. The former equation is the same as $z_{11}+ z_{23} = z_{13}+z_{21}$, which is also equivalent to c=0. Indeed, all of the coefficients of these two equations , on both the left and the right side will be equal to 0 when c=0. ***It follows that these equations will be inconsistent only when some of these coefficients are formally equal to 0 while others are not.***

It can easily be shown that the situation Δ = 0 leads to the same result.

EXAMPLE:

Let $K =$

y/x	z_1	z_2	z_3
z_1	z_1	z_2	z_2
z_2	z_2	z_1	z_2
z_3	z_1	z_1	z_3

Notice the presence of the subalgebra G = {z_1 , z_2 } in this table. Substituting for the c-relations in equations (1) and (2) we find that:

$$Equation[1]:$$

$$(i) u + z_{33} - L = z_{12} + z_{21} + z_{33} - z_{12} - z_{23} - z_{31}$$

$$= z_2 + z_3 - z_2 - z_1 = z_3 - z_1$$

$$(ii) v + z_{11} - L = \ldots = z_1 - z_2$$

$$(ii) z_{12} + z_{21} - (z_{22} + z_{11}) = \ldots = z_3 - z_2$$

$$\therefore (z_3 - z_1)\alpha + (z_1 - z_2)\beta = z_3 - z_2$$

Dividing through by z_2 reduces this equation to:

$$\alpha(\beta - \alpha) + (\alpha - 1)\beta = \beta - 1, or$$

$$\underline{\alpha^2 - 2\alpha\beta + 2\beta - 1 = 0}$$

The second equation is given by:

$$Equation[1]:$$

$$(i) w + z_{22} - L = z_{13} + z_{31} + z_{22} - z_{12} - z_{23} - z_{31}$$

$$= z_2 + z_1 + z_1 - 2z_2 - z_1 = z_1 - z_2$$

$$(ii) u - (z_{11} + z_{22}) = \ldots = 2(z_2 - z_1)$$

$$(ii) L - v - z_{11} = \ldots = z_2 - z_1$$

$$\therefore (z_1 - z_2)\alpha + 2(z_2 - z_1)\beta = z_2 - z_1$$

Dividing through by z_2 reduces this equation to:

$$\underline{(ii) \quad \alpha = 2\beta - 1}$$

This pair of equations has two solutions, corresponding to two kinds of surfaces that can hold the algebra K. The first is given by $\beta = 1/2, \alpha = 0$. In this algebra z_1=0, and z_2=2z_3.

If $2\beta - 1 \neq 0$, then Equation 1 factors out to an identity, and we are left with the relation in Equation 2. Substituting in the expressions for a and b in terms of the roots, this becomes, finally :

$$2z_3 - z_1 - z_2 = 0, or$$
$$z_3 = (z_1 + z_2)/2$$

❋❋❋❋❋❋❋❋❋❋❋❋
❋❋❋❋❋❋❋❋❋❋❋❋

Roy Lisker

1994-95; Revised 1999

#103...