On the constituents of mathematical models in physics

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Starting with the pre-Socratic philosophers of Miletus and Elea, physicists have always employed a relatively small number of "pictures" to guide them. Aristotle and Kant raised these mental images to the level of Categories. When confronted with a physical phenomenon, it is natural to grope around for a kind of mathematics that will adequately represent the constructs and images. There is no guarantee that anything can be found, but it happens often enough to be known as the *"unreasonable effectiveness of mathematics"* . In this article I will set up and treat, item by item, a list of the most frequently invoked of these fundamental images. This will be followed by an investigation into the forms of mathematics (if any) that is most appropriate for them.

All of these fundamental images embody A Priori assumptions:

(1) Either in their choice of mathematics; or

(2) Through the invention and development of a mathematics determined by the requirements of the model.

These assumptions may also be present in a prior stage, in the selection

process on the data itself, in the "triage" on erroneous or irrelevant observations, to the benefit of others deemed more interesting or important ones. The most famous historical instance of this process is that of the selectiveness practiced by Mr. Arthur Eddington on the photographs he showed to the press to 'prove' General Relativity through observations on the bending of light during a solar eclipse.

(3) Some A Priori assumptions may find a natural representation in mathematics, but there are also those for which there may not be any simple representation by mathematics, so-called 'non-quantifiable' phenomena.

(4) Finally there may be mathematical concepts, of great simplicity, which do not find there correlatives in the physical universe. I am thinking in particular of the "Euclidean point". This both *does and does not* exist in nature, depending on the physicist one is talking to, or how a given theory is interpreted. We will spend some considerable time discussing this in what follows.

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A short list of the basic constituents used in physical models

(a) Hard Spheres

(b) Conservation Laws

(c) Harmonic Oscillators

(d) Euclidean Points

(e) Collisions

(f) Dependent and independent variables

(g) Action at a distance

(h) Probabilities

(i) Fields

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(a) Hard Spheres

Despite the success of the field concept, physicists continue to model certain collision phenomena, whether in the microcosm, modocosm [1] or macrocosm (stars, asteroids, galaxies) in terms of the elastic or inelastic collisions of hard spheres. It is only in the microcosm, at the level of elementary particles, that one can replace this intuitively compelling image by force fields and probabilities. These treat the specifics of the phenomenon, but do not address the basic issues. Following Democritus matter, at some level, must be impenetrable.

The attributes of "hard spheres" include:

(i) They have 3-dimensional volume

(ii) They are compact. This is not incompatible with, as in gravitation, a filigree of tentacles reaching into space mediating "action at a distance" (A modocosm compromise between the hard sphere and the field).

(iii) They have a well defined "inside and outside". This is interesting and brings in some fairly advanced mathematics. The Jordan Curve Theorem states that any simple closed curve in the plane (closed surface in space) has a well defined inside and outside, separated by a boundary. Thus, for a hard

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[1] **The in-between cosmos of daily life**

sphere to separate inside from outside it must have a boundary.

Yet boundaries have become increasingly unwelcome in modern physics. Diffusion is everywhere. The 2nd Law of Thermodynamics, though based on classical principles, guarantees that energy will dissipate over time, and ultimately over the entire universe.

The picture of the “well defined boundary” is mostly clearly violated by the phenomenon of quantum tunneling. What people don't seem to realize is that Schrödinger’s Cat tunneled out of its box a long time ago, which means that we can see for ourselves if it is alive or dead! Even the Black Hole, the most perfect container in theoretical physics, will dissipate entirely through Hawking radiation if one waits long enough.

How does one model a physical space without boundaries or containers? One thing one can do is replace a 'particle description' by a 'wave description' as is done in the DeBroglie interpretation of standard quantum theory. Waves are everywhere at once; geometry is no longer Euclidean but 'distributive', that is to say, subject to the probability distributions of wave functions.

(iv) Normally, hard spheres should have a well defined density. Yet from the 18th century onwards, this inconvenience has been evaded by contracting material object to points for the purposes of calculation.
Such an entity may be unpicturable for intuition, but it must be acceptable as mathematics, which is not always the case: the "point" becomes a singularity, an exceptional point that must be cut out of the description.

In general, dancing around between points, particles and waves is

standard physics for the last 300 years, and as long as it gives correct predictions it doesn't matter if the composite picture makes no sense at all.

The work of cleaning up the chaos is proper to philosophers, and as long as one continues to make better refrigerators, one doesn't have to bother with them.

(v) Their mass is conserved

Let us agree, for the moment, that these are the 5 essential attributes of hard spheres. They permit a wide class of variant models with "secondary attributes". A good point, because the ancient distinction between primary and secondary characteristics has its roots in the fundamental properties of the most basic images employed in physical descriptions.

(1) Hard spheres can be allowed to change their shape, provided their mass is invariant and they remain homeomorphic to a sphere. (Elasticity)

(2) The density can be uniform or variable within their boundaries. At the far extreme, the solid mass can be softened to a "liquid drop" as in the Millikan charge experiment.

(c) Or a perpetually fluctuating density, provided that the total mass remains constant

(c) They need not be spheres, but can assume the various forms of 3-manifolds: sphere, tori with one or more holes, knots."Hardness" simply means that they do not change their topology in the course of an interaction

(d) They are normally treated as indecomposable, but they can also be treated as decomposable, provided that the system of particles into

which they are split obey the conservation laws and, therefore, operate as a single system.

Whatever the nature of the colliding objects, they must obey:

(i) Conservation of Matter

(ii) Conservation of Momentum

(iii) Conservation of Energy

They also obey the principles of Newtonian mechanics:

(1) Inertia

(2)Action equals reaction.

(3) Force equals mass times acceleration. Because of these principles all interactions of 'hard spheres' in a Newtonian universe resolve into motion of the center of gravity, and motion around the center. The CG itself is an example of a singular dimensionless point.

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(b) Conservation laws.

The proper mathematics for dealing with the conservation laws of physics was created by Emmy Noether in 1918, who established the relationship between the conservation laws and the fundamental symmetries of nature. Provided one has some way of defining an Euler-Lagrange variational integral one can extend these laws to general and even to abstract spaces, as is done in String Theory.

The conservation laws, in other words, are a natural consequence of something more fundamental, the variational principles from which geodesics

can be derived. A Priori principles can be enunciated globally, but they really have to have an application locally. In the 17th and 18th centuries various global principles were enunciated, such as the "Principle of Least Time (Fermat), "Principle of Sufficient Reason (Leibniz)", "Principle of Least Action (Maupertius and Euler)". Yet it is wiser to stay within a local context. Physics, as a science is restricted to what is observable, and what one should be looking for are one really needs are direct observations that some things in the *accessible* universe are connected to others.

For example: an astronomer observes the light coming from a quasar several billion light years away. The observation of this light gives him what he needs for his PhD thesis, he can join the faculty and ultimately acquire tenure. A light signal that started out 8 billion years ago so affected his life that it gave him life-long security! *That* is inter-connectedness, and it depends on the time delays involved in making such connections.

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(c) Harmonic Oscillators

In the tool kits of the model builders of physics, no instruments are employed more often than harmonic oscillators. They are used for modeling just about everything, from the vibrations of the strings of Pythagorus's monochord, Maxwell's equations, Planck's quanta, Boltzmann's aggregates of rebounding molecules, Fourier's theory of heat conduction, the lumineferous ether, the Bohr atom, DeBroglie's waves, Schrodinger's waves....

Fundamental to the harmonic oscillator is periodicity. Since physics is

based on measurements, the identification of the presence of a periodic system depends upon readings from clocks, signaling the return to an initial state. This procedure is somewhat circular: the periodicity *of a harmonic oscillator* depends on the existence of a *machine* that identifies its presence. The functioning of this machine, a clock, depends in its turn on the *laws of universe mechanics* : Newton, Maxwell, Boltzmann, Schrodinger, etc.

The requirements for the re-occurrence of causal cycles demand nothing less than a total reformulation of the standard paradigm of causation. In a "harmonic oscillator" there has to be something that oscillates. In fact, it can be any observable, Momentum, Spin, Energy, Location. It is only necessary that the oscillations take place in time. One could, for example, imagine a particle going around and around in the same circular path, yet in wildly different time intervals. This would not be considered a harmonic oscillator. It is the oscillation in regular *temporal* periods, congruent at all corresponding points that is required.

This is translated, via the paradigm of standard causation, into the requirement that a system returns to its initial state. Total determinism then asserts that it will oscillate in equal time intervals throughout past and future eternity. Representations by analytic functions are replaced by Fourier Series built from sines and cosines with the same periods, $2\pi/L$.

This change of emphasis is necessary but not sufficient; Fourier Series can also represent functions with jumps. Indeed, the computations of the coefficients of a Fourier series are not done in the neighborhood of an *instant,*

but as integrals over *the entire length of a period*. We are therefore speaking of *a very different species* of causation, namely *harmonic causation*:

Harmonic Causation

Predicting the behavior of a harmonic system depends upon information present within an entire period, *plus* the observation that an initial state is exactly reproduced at the beginning and end of the period. The standard paradigm then guarantees that the behavior within that period will then be exactly reproduced through eternity (or in such time as …) in both past reconstruction and future behavior. For practical purposes, this can be modeled by a Fourier series.

Discontinuous jumps are not inconsistent with causality, but they do pose special problems. Collisions also produce discontinuous jumps in velocity, momentum and energy, yet are readily amenable to a causal rationale.

When it comes to applications to real experiments, one finds that harmonicity is a strong *A Priori* assumption not always inherent in the data itself. One has to determine that the clock which signals the period *is itself* a periodic system. Examination of its behavior, whether continuous, differentiable or analytic, may reveal the presence of jumps. Coming up with an explanation for them brings in the possibility of a *dual or parallel process*.

This is a model in the form of an infinite series which captures the essence of a situation in which manifest behavior is continuous, even differentiable, but the representation breaks down at key points, represented by the jumps.

Such phenomena can be modeled by *infinite series of conditionally convergent functions*. In particular, any Fourier series that models a jump discontinuity must be conditionally, not absolutely convergent. There is a theorem stating that any *absolutely* convergent infinite series of functions of a variable t, which has a value v at some point T, must converge to v as t → T.

One could even speak of a secondary time dimension s, in addition to the "surface time" t, which measures the progress of a visible process. s measures "how far out must one go in the series before it begins to converge", or equivalently, "before it diverges away from the 'jump value' v, at time T". This can be made mathematically precise: Let

$$\boxed{F(t) = \sum_{n=0}^{\infty} g_n(t)}$$

All the functions, g_n are differentiable or analytic, and the series converges everywhere. However, at a certain point T, the continuous limit as t→T is u, but the series itself converges to v larger (say) than u. Take a point t_1 close to T, and define N as follows:

Definition: If the partial sum to N is given as $\boxed{F_N(t_1) = \sum_{n=0}^{N} g_n(t_1)}$ ***, then***

$$\boxed{\left|u - F_K(t_1)\right| < \tfrac{1}{2}\left|v - u\right|}$$ ***for all K > N.***

$N(t_1)$ is the "second time dimension" s of the "parallel process" that moves along, invisibly, in the infinite series itself. s becomes infinite at the time t= T. This is what is meant by the "internal time dimension". It

provides a model for deterministic causation (relative to the causal dimension s) that allows for discontinuous jumps in an oscillating quantity.

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(d) Euclidean or Dimensionless Points

"The typology of Grothendieck is incredibly complex. Like Gauss, Riemann, and so many other mathematicians, his major obsession was with the idea of space. But Grothendieck's originality was to deepen the idea of a geometric point." Pierre Cartier

The physics community both rejects and accepts the existence of dimensionless points in real space-time. It's been more than 2 millennia since Democritus presented elementary arguments to show that material objects cannot be fragmented or decomposed into ideal Euclidean points, otherwise "matter" itself disappears. Physics is based on measurements, and if a recorded measurement is a "0", one then says that whatever one is measuring is absent. If a particle of matter is decomposed to a Euclidean point, then the mass measurement of that point will be 0.

On the other hand, physicists tend to assume that there are many things that exist *at* real "point locations" (and "point instants"), such as: centers of gravity; centers of infinite gravitational potential; sources of force fields; Black Holes.

(a) When two objects collide, the surface of collision is at most two-dimensional; otherwise there is interpenetration, and one is no longer speaking of a solid body.

(b) The transfer of 'identity', the 'effect' on an object O, which transforms its 'identity' from I_1 to I_2, under the action of a cause C, is

assumed to be effected instantly, at precise locations in space.

(c) Electrons are "point sources" of charge.

(d) Quanta, travel along “pure" 1-dimensional lines in space-time.

***Question:* Do the bodies of quanta fill up real space? Imagine a burst of light emanating from a source at location L. After k seconds the quanta from the first burst of illumination fill up the surface of a sphere at a distance of k light-seconds. The area of this surface is finite, and the number of quanta must be finite, because the energy on that surface must be finite, and E= nhν, where ν is the frequency. Therefore, divide the surface area by the number of quantum and one obtains the surface area of a quantum. One can do the same for the entire sphere of light emerging from the source, to obtain the volume of a quantum. Or one can say that the frequency goes down, which may be a way of saying the same thing.**

Of course the question: "Do real Euclidean points exist in real Euclidean 3-space?" is of real interest only to academic philosophers, pedants and casuists; it is simply our intention here to point out that physicists talk, sometimes this way, sometimes that.

The "core of the Black Hole", into which all the matter disappears, is treated in the literature as a *pure Euclidean point*. The question of whether or not such an object makes any sense in a real universe is side-stepped directly, but surfaces indirectly in the two possible interpretations:

(a) The Black Hole is a singular point *of* the space-time manifold, at which the density of matter goes to infinity. Thus, the pure Euclidean point is

coupled with a *real infinity* in our universe.

(b) One might instead cut the locus of the Black Hole out of "real" space-time. Space-time then continues to be free of singularities. This may make for some topological problems related to connectedness and continuity.

(2) Field sources for which the potential goes to infinity at a specific point. For example, if two massive particles accelerating towards each other under the effects of gravity were to experience no resistance in their encounter, their accelerations would go to infinity at their mutual center of gravity. And the same is true for the encounter of charged particles of opposing signs. These examples are related to that of the Black Hole, which collapses into the point source of its own gravitational potential.

When speaking of *actual* particles, one must work with an object that takes up a finite region of space. The concept of an "instant" without extension is also without merit; time is measured by clocks, and clocks need a minimum duration for their periodic cycles.

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(e) Collisions.

Collisions are jumps: instantaneous causal singularities . One can model them by Dirac delta-functions, intervening in an otherwise basically smooth description of inertial systems in isolation. They can also be represented as a variety, the solution space of velocities, for the pair of equations of the conservation of momentum and the conservation of energy.

The relevant issue is whether or not Inertia should be considered a force

akin to gravitation, dark energy, inflation and so on, and if so, whether this force is released by the abrupt change of velocities of two objects at the moment of mutual impact. Newton asserted that force is equal to mass times acceleration, and one cannot deny that an 'acceleration' is present at any change in velocity.

Question: **Since gravitational and inertial mass are equivalent, it should follow that the Equivalence Principle of *General Relativity* applies to inertia as well as to gravity. That is to say, it should be possible to describe collisions in two ways: either as an instantaneous burst of acceleration, or the presence of a gravitational field. Has anyone ever done this?**

From the standpoint of *Special Relativity* the release of energy upon collision may entail the conversion of a minute amount of mass into energy: $E = mc^2$. Such a conversion would be impossible to detect in the extremely minute collisions one experiences on earth, but it is possible that one might actually measure such a mass loss in the collisions of entire galaxies that have been observed in outer space.

Furthermore, if, in a collision, the velocity of an object of mass M changes from v to u, there will certainly be a detectable change in the relativistic mass, from $\boxed{M_v = \frac{M}{\sqrt{1 - v^2/c^2}}}$ **to** $\boxed{M_u = \frac{M}{\sqrt{1 - u^2/c^2}}}$ **. These arguments are purely speculative, but are worth looking into. If the Newtonian law, "Force equals mass times acceleration" is to apply to the whole universe, it seems incorrect to exempt collisions.**

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(f) Dependent and Independent variables:

Functional dependence may be a simple matter of definition in mathematics where, for the most part one can assume the application of the Inverse and Implicit Function theorems.

In the language of physics the distinction becomes important when one wants to establish the relative independence of the various "dimensions" so beloved of the string theorists, or the "observables" of quantum theory. Thus, "observables" (energy, spin, momentum, location) are considered to be "dependent" on the "parameters" (time and mass).

In physics there are always hierarchies of dependency.

$$\begin{array}{l}\textbf{Energy} \rightarrow \textbf{3D location} \rightarrow \textbf{Time}\\ \downarrow \\ \textbf{Mass}\end{array}$$

Energy is a dependent variable, either directly on time, or indirectly on the dependence of location on time:

$$\boxed{E = E(m,x,y,z) = h\nu = \tfrac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right)}$$

To mix up independent and independent variables is to mix up observables and dimensions. Although our experience of time comes to us uniquely in the form of a *discrete* series of events, by raising it to the level of an autonomous dimension, time itself is treated as a *continuum*.

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(g) Action at a distance

In the 17th century, action at a distance was considered a logical impossibility. This was the principal argument of the Cartesians against the Newtonians, when the presence of action at a distance was portrayed as an integral assumption of Newton's *Principia* . The notion was abandoned once again when Einstein's theory of General Relativity had as a consequence the impossibility of any signal, including the gravitational field, to travel at a speed greater than light. Yet once again the concept was revived, very recently (relative to the history of science) in the theorems of J.S. Bell and their confirmation in Aspect's experiments.

Action at a distance in its pure form is a paradigmatic example of the phenomenon of non-locality. What one is seeing in quantum entanglement is a new way of blurring the line between matter and space, normally considered independent parameters or dimensions. One sees traces of this in the thinking of Leibniz, which maintained that space is not a real observable, but rather expressed a relationship between pieces of matter.

That matter may be a manifestation of space, (an inversion of Leibniz) is inherent in various interpretations of General Relativity, which derive matter entirely from the curvature of space, so that in fact, matter is a "symptom' if you like, of universal geometry.

It would be interesting to bring together these two ways in which "space" and "matter" are confounded, namely "entanglement" and "curvature of

space time". I am thinking in particular, of "loops", whereby a pair of entangled particles "meet" at a single location, with signals going around the loop at the speed of light.

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(h) Probabilities

The concept of probability as it is used in physics is actually rather different from the one that is standard in mathematics, or from the way in which it is used in statistics. When a physicist working in Statistical Mechanics, for example, says that "the chances of the occurrence of p are 5%", he means the following: the normalized time integral of the number of occurrences of p, from minus infinite time to plus infinite time, is 1/20. In other words, if one waits long enough, the time average of any random variable *must* even out to the known probability.

Its' uses in quantum mechanics or elementary particle theory are similar. When a nuclear chemist states that radium, for example, has a "half life" of h, the chemist means that, over a sufficiently long period of time, the decay rate must become the exact value of the calculated half life.

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(i) Fields

Much ink has been spilled on the ontology of fields. Force fields impose a fiber bundle structure on space or space-time. They are connected topological manifolds structured by a gauge group. They replace the more philosophically problematic images of 'hard spheres', 'density at a point' and

'collisions', but bring in new problems of their own. Most commonly, a field emanates from a point source, and is expressed by a potential function that goes to infinity at that point. In the case of gravitation there is a peculiarity in that the field *emerges* from a massive source, *travels long* distances at the speed of light, then *pulls* every object in the expanding force sphere *towards itself*. How can a phenomenon both push *away* from a source, yet also draw everything to it? Gravity is a *pulling force* that *pushes away from its source*! What is the motive force that 'pushes' the 'pulling field' outwards? Einstein describes it as a kind of dynamic geometry that is created second-by-second by all the matter in the universe.

One of the signal attributes of a field is that it is invisible until a "test particle" is dropped into it. The particle cannot be any lump of matter, but must have some affinity with the field. Thus, a neutron will swim through an electro-magnetic field with no ill effects; but a proton or electron will immediately be coerced into motion. Does the field exist in fact, if there is nothing moving within it? This is a real question with respect to gravitation, because it is normally described as an attraction *between* masses. If a massive object is isolated and therefore has nothing to attract or be attracted by it, is there a field? We rediscover non-locality within General Relativity itself, with no need to appeal to quantum theory.

Thus the test particle is itself a field, and the experiment of dropping a test particle into a field is really an interaction between two fields, or two manifestations of the same field.

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