Ordinal Maps
Ordinal Arithmetic and Dynamical Systems

Roy Lisker
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The space F of all real functions on the real line f: x —> y =f(x) is obdurate to universal structures. Normally one selects certain "tame" sub-spaces, the differentiable or continuous functions, measurable functions, Sobelev spaces, and so on.

In this exercise I will be proposing a way of associating a "transfinite index" to every point (x, f(x)) on the graph of a very general sub-space of F which is capable of being extended with little difficulty to most of the other elements of F. The useful employment, however, of this method lies primarily in the analysis of places where the functions of F become unbounded.

The first restriction of F is to a space F^+_0, defined as follows:

(a) The domain of f ε F^+_0 is the complete non-negative x-axis,

$$R_0^+ : 0 \leq x < \infty$$

(b) The range of f is a subset of the non-negative y-axis

(c) For all f in F^+_0 , f (0) = 0

It is difficult to imagine that one could find any universal structures in such a vast space. Certainly topologies, tubular neighborhoods, algebra, manifold structures, composition identities are hard to come by.

Of course, $f, g \in F_0^+ \rightarrow f(g) \in F_0^+$, but this is not much to go on. However, a simple transformation puts the functions of F^+_0 into a 1-1 correspondence with a subspace B, whose functions

incorporate a natural dynamical system that embodies structural information:

Definition: **The reductive operator L over F^{+}_{0} is defined by:**

$$L(f) = x\frac{f}{(1+f)} = \varphi(x)$$

Properties of L :

(i) φ (x) is defined at every point of the domain of f

(ii) φ (x) <x for all x > 0

(iii) If f is unbounded at some point t >0 - that is to say, arbitrary large values of f are found in any neighborhood of t, then there are correspondingly, points of φ (x) in the neighborhood of t arbitrarily close to the fixed point axis, I (y=x) .

(iv) L is invertible.

(a) At x=0, both f and φ = 0

(b) For x >0, one easily finds the inverse solution

The application of the operator L to the space F^{+}_{0} produces a function space B characterized by the above list of properties.

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Let φ be an element in B. Then, clearly

$$x > \varphi(x) > \varphi(\varphi(x)) > \ldots > \varphi^{(n)}(x)$$

Theorem:

For given x, φ ε B, the iterative limit $\lim_{n\to\varpi} \varphi^{(n)}(x)$

always exists = δ 0.

Observe that from the 3 symbols for somewhat different versions of infinity, (, , $Aleph_0$) the ordinal symbol is selected.

The theorem is all but self evident, because all members of B are monotonically descending and bounded from below by 0. Observe also that if δ > 0, then the value of φ(δ) at δ is not δ itself but must drop down to a lower value, x' < δ. In other words, the orbit of φ(x) moves to a point on the fixed point axis I, but φ drops down at that point to some other value.

If x′= 0, the iterative process comes to a halt. Such a value for φ ( ) corresponds to a zero in the associated function in F^{+}_{0} ,

$$\Lambda(x) = \frac{\varphi}{x-\varphi}$$

It will not be of interest to us, in terms of the structures we are examining, for there to be 0's in either of these two functions except at the origin.

Definition: If $\lim_{n \to \varpi} \varphi^{(n)}(x) = \delta$ we write $\varphi^{(\varpi)}(x) = \delta$

Although ω is the first countable ordinal this notation can clearly be extended to all countable ordinals.

Definition: Let γ be any transfinite countable ordinal. Then we write: $\lim_{n \to \gamma} \varphi^{(n)}(x) = \varphi^{(\gamma)}(x)$, where n runs through all finite and countable ordinals up to and including .

Theorem: This notation is consistent and free from contradiction. Proof by transfinite induction.

Theorem: Let φ ε B*. For any x ε R^{+}, there is a transfinite ordinal Γ (x, φ), such that $\varphi^{(\Gamma(x,\varphi))}(x) = 0$

Informally: if the iterative system moves to an attractor δ, and the value of φ(δ) is less than δ, one can continue to apply φ to δ , moving to the next attractor , $\delta_1 = \varphi(\omega)(\delta)$, $\varphi(\omega)(\delta_1) < \delta_1$.

This process can be repeated indefinitely; as long as the system hasn't yet arrived at 0, a descending chain of attractors is set up, which cannot come to a halt at any limit point greater than 0.

By Zermelo's Well-Ordering Theorem this descending chain must eventually arrive at 0 in a number of steps given by some ordinal Γ.

Theorem: There exists a minimal countable ordinal $i(x, \varphi)$, such that $\varphi^{(i(x,\varphi))}(x) = 0$.

That such an ordinal exists follows from the Well-Ordering Theorem. That it must be countable can be seen from the Lesbesgue Integral Theorem. The intervals between successive attractors must have some positive length, but the sum of uncountably many positive lengths cannot be equal to the finite length $[0,x]$

We will call the number i the ordinal index of x in φ .

Theorem: The ordinal index of any function in B^* is always a countable transfinite ordinal without finite part.

Since there are no zeros of φ save at the origin, no finite iteration from any point $x > 0$ can reach 0.

Next, let ϕ be any function of F^+_0. If ϕ is bounded in any finite sub-interval of R^+, then $L\phi = \lambda$ will be bounded away from the fixed point axis in such a way that it will hold no attractors other than 0. In this case there will be only one ordinal index for all positive points of the domain, namely ω.

In particular, if ϕ is already a member of B^*, then $L\phi$ will be reduced to a function whose points all have the "default" index ω.

This suggests a further restriction on the space of all functions, to a space F', of functions which have some unbounded values in the finite part of their range. The correspondence allows us to associate the transfinite ordinal structure of the functions in B* with the unbounded points of the functions in F' . This provides structural information at these points that may not be apparent with functions in F′.

For example, one sees that the graphs of functions in B* will, in general, have limit points on the fixed point axis I, of two types: topological limit points and iterative limit points. The topological points are not the endpoints of any dynamical orbit, but are simply limit points of the 2-D set of points $(x, \varphi(x))$.

This distinction between orbit end points and topological limit points is not apparent in the structure of the unbounded points of the corresponding function in F' , but is revealed when moving to B* .

Furthermore, every attractor will have orbits of various countable transfinite ordinals associated with it. This indexing of attractors in accordance with ordinals cannot be seen in F, but is revealed under the transformation L.

Thus, L behaves somewhat like a Fourier Transform that reveals the structure of a given function in terms of its harmonics.

A Model from Number Theory

If one restricts the functions of B to those whose iterates on the domain of rational numbers Q converge to rationals, then all of the above considerations will apply if the domain R^+_0, is replaced by the domain Q^+_0 of positive rational numbers.

Let r be a positive rational. It can be written as a fraction in lowest terms, r = p/q, p , q integers. Let P stand for the "numerator function"

$$P(r) = P(p,q) = p \,,\ (r = p/q \ \varepsilon\ Q^+_0)$$

P has the following properties:

(1) P associates a unique integer with every rational r

(2) P is "loosely monotonic" in the following sense: If r > n, where n is some integer, then P(r) > n . The converse is not true P(r) < n tells us nothing about the magnitude of r.

(3) In fact, although it is never infinite P is *unbounded* in the neighborhood of *every point* of its range.

Let us now apply the reductive operator L(P) defined above as

$$L(f) = x\frac{f}{(1+f)} = l(x)$$

to the numerator function P . The result is:

$$L(Pp(r)) = r\frac{p}{(1+p)} = \frac{p}{q}\frac{p}{(1+p)}$$

$$= \frac{p^2}{q(1+p)} = \varphi(r)$$

The function φ associates a dynamical system with P that is of great interest. A simple calculation shows :

$$\varphi^{(2)}(r) = r\frac{p^3}{(1+p)(1+p^2)}$$
$$= r\frac{1}{(1+\frac{1}{p})(1+\frac{1}{p^2})}$$

Continuing by induction one has:

$$\varphi^{(n)}(r) = r\frac{1}{(1+\frac{1}{p})(1+\frac{1}{p^2})(1+\frac{1}{p^4})\ldots(1+\frac{1}{p^{2n-1}})}$$

It turns out to be remarkably simple to evaluate the limit of this expression as n goes to infinity. If one multiplies together all the terms in the above product, one ends up with exponents of 1/p that are the binary representations of all the integers from 1 to 2^n ! Therefore

$$\varphi^{(\varpi)}(r) = r\frac{1}{(\sum_{j=0}^{\infty}\frac{1}{p^j})} = r\frac{1}{\left(\frac{1}{1-\frac{1}{p}}\right)}$$
$$= \frac{p}{q}\frac{p-1}{p} = \frac{p-1}{q} \; (!)$$

Thus, the first ω -cycle of iterates of j(r) reduces the numerator by 1. Setting r' =(p-1)/q one recognizes the new rational argument r' may not be in lowest terms. Thus r' = p'/q' , with p' $\leq$ p , q'$\leq$ q . If φ is applied to this number the result will of course be strictly less than r' .

$$\varphi(r') = \frac{p'^2}{q'(1+p')}$$

Iterating this ω times produces a limit result of(p'-1)/q' . Of course if p' = 1 , the process terminates. In general, given any rational r, there will be a finite number of ω-cycles of iterates on φ which sends the orbit of φ from r to 0. This is the transfinite countable index i (φ , r) .

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Let review some definitions and establish others:

Definition 0: **The expression "countable transfinite ordinal" will be abbreviated c.t.o.**

Definition 1 **: If φ is a function in B , x a point in the range R^+ , then i(φ ,x) , called the *ordinal index of the point x , in φ* , is the smallest countable transfinite ordinal λ , such that $\varphi^{(\lambda)}(x) = 0$.**

Definition 2: If φ is a function in B, then j (φ , x), called the *ordinal index of the function φ,* is the limsup, or least upper bound of the ordinal indexes for all points in the range of φ .

One easily sees that there are two kinds of functions: those in which the ordinal index of the function is achieved on a subset S of points in the range, and those in which the function index is only the upper limit of all the point indexes and is never realized at any specific point of the range.

One can construct examples of functions of both types, for any given countable ordinal λ . The number theoretic function P described above has an ordinal index of ω^2, but this is not attained

at any point of the domain. If, by convention, one incorporates 0 into the domain of every function of B, then explicit constructions can be done of functions which attain any countable ordinal at 0.

Here is a standard construction for doing so. We consider a class of functions $\mathbf{A} = \{A\lambda\}$, where λ is some given c.t.o..

Let λ be chosen. Let $Q_{[0,1]}$ be the collection of rational numbers in the interval [0.1]. We construct a topological representation of λ by selecting out a sequence of elements of S which mimic the sequence of ordinal numbers converging to λ . $S = 1 > r_1 > r_2 > ...... > r_n > ....$, converging to 0, and to various limit points in Q. It is a well-known result that all c.t.o's may be represented in this way. Indeed, any order type of countable cardinal can be given such a representation.

The function $A\lambda$ is now defined as follows:

(1) $A\lambda\ (r_k) = r_{k+1}$.

(2) If $r_k > x > r_{k+1}$, then $A\lambda\ (x) = r_{k+1}$.

In order to represent all c.t.o.'s in this fashion, one must admit functions φ which drop down to 0 before reaching the point x=0, but this presents no difficulty.

The functions $A\lambda$ have the property that the function index λ is explicitly attained at the point x=1. However, by constructing a function that attains its function index as a limsup only, and assuming the Axiom of Choice, it is even possible to have as function ordinal the first uncountable ordinal ω_1 !

The construction proceeds as follows: A construction due to Vitali decomposes the interval [0,1] into a collection of uncountably many congruent countable sets $\{ C\gamma \}$, everywhere

dense in Q , where γ runs over the Hamel basis . For each γ one chooses a different c.t.o. $\lambda = f(\gamma)$. Since the number of c.t.o's is Aleph1 , the Axiom of Choice allows that there is some way of doing this.

The functions $A^*\lambda$ corresponding to each l, are readily constructed in a manner similar to that for the functions $A\lambda$, above. Indeed, one maps Q isomorphically onto each $C\gamma$, guaranteeing that the subsequences S will go into representation sets for the ordinals λ. The intermediate points x in each $C\gamma$ will be sent by $A^*\lambda$ into the next representation point on the left.

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The Dual collection B*

The functions in the collection B will converge to 0 from any point x under sequential iterations up to λ , the c.t.o. index of x. This suggests a natural correspondence with the dual collection B*, of functions ρ defined on R^+ with the property $\rho(x) > x$ for all x in R^+ . There are many ways of setting up this correspondence. This one is based on the natural dualization of the function P(r) = p, r = p/q, namely Q(r) = q. Define the operator L* on any function in **F** by

$$L^*(f) = x\frac{1+f(1/x)}{(f(1/x)} = \rho(x)$$

When L* is applied to the function Q, one obtains the associated dynamical system

$$\rho(r) = \frac{p(q+1)}{q^2}$$

This is dual to the corresponding dynamical system for P(r) =p, and converges, after iterations, to the value r′= p/(q-1) = p′/q′ in lowest terms.

The dual systems in B and B* are completely equivalent. Since one is setting the function f in F to 0 at 0, this becomes the value of f(1/x) at x = ∞. Thus, a system in B* "loops back" to the origin when it reaches the end of its c.t.o. function index of iterations.

Contracting functions in the Complex Plane

This analysis no longer works for the class G of functions g (z) in the Complex Plane, C defined by the property $|g(z)| < |z|$.

Starting from any point z_0 in C, the application of iterations to g produces a *set* $H = \{g^{(\omega)}(z)\}$, a sub-set of the arc of a circle with radius R, the modulus of any member of H.

Iteration from ω to 2ω can produce virtually any closed subset in the interior and boundary of the circle $|z| = R$. The analysis of the properties of a c.t.o. index for complex functions requires a separate study, outside the scope of this paper.

Invariant c.t.o. structures on the Space FM of all functions f: N --> N, N = [-M, M], M>0

The discussion in the preceding pages can be readily generalized so as to give some insight into the c.t.o. index characteristics of *all functions y=f(x),* with domain = range = N= [-M, M} for some specified bound M.

f here stands for any function whatsoever in this class. Let (x_0, y_0) be some point on the graph of f. Since N is compact, the collection of iterates $\{f^{(n)}(x_0)\} = \{x_n\} = X_f$ must contain at least one limit point in its closure that is in the interval [-M, M]. Let $S= S(x_0)$ be this collection of limit points.

Since f can be any function, it need not be continuous anywhere. For obvious reasons, one can define

$$f^{\omega}(x_0) = f(S) = S_1 ,$$

the set obtained by applying f to every element of S. Clearly this process can be continued another ω times, leading to a set $S_2 = f^{2\omega}(x_0)$, and so on indefinitely.

Theorems:

Theorem I : Given the above function f and N = range/domain = [-M, M] as above, there is a c.t.o. λ such that $f^{(\lambda)}(N) = C = f(C)$ is an invariant domain in N.

By the method of defining $f^{(\lambda)}(N)$ at the limit ordinals, all of these sets S are closed. Therefore the sets T = N- S (with the endpoints of N removed), are open. Open sets contain a countable number of rational points, each surrounded by a countable number of open neighborhoods contained in T, defined by circles with rational radii.

It follows that each application of $f^{(\omega)}(S)$ removes only a countable number of rational points from T, which can be surrounded once again by a countable number of circles with rational radii, contained in T. Such a process must end either in an invariant set C, or in the Null Set, in a number of operations given

by some countable transfinite ordinal, or c.t.o. . (It would be strange indeed, if *uncountably* many steps were required to exhaust a countable set of points and neighborhoods!)

Theorem II. Let x_0 be a point in the range N = [-M, M]. Once again, define $f^{(\omega)}(x_0)$ as the set $s(x_0\ )$ of *limit points* of $L=\{f^{(n)}(x_0)\}$, and

f $(s(x_0)$, as an application of f to the entire set s, etc. Continue this process to exhaustion. Let S_λ be the *union* of all the sets generated by this iterative process, up to the c.t.o. λ.

Then there exists a c.t.o. γ such that f (S_γ) is contained in S_γ , that is to say, S_γ contains all the points of N which the process of iteration of f on x_0 will eventually reach.

The proof is similar to that of Theorem 1, with N being replaced by the invariant set C, from which, once again, a countable number of rational points are removed from C- S_λ at each stage of the process.

(The following theorem is incomplete. There are reasons to believe that it is true, but what is presented here is at best a sketch.)

Theorem III. Given f , x_0 , S_γ , there is at least one *minimal*

***invariant set I (f, x_0)* contained in S_γ. This has the property that $f(I(f, x_0)) = I(f, x_0)$.**

One modifies the proof of Theorem I, to cover closed subsets of a closed set. The details can be added later.

***Minimal invariant sets are cycles, and the equivalent of fixed points for the full generality of functions on a real interval.* They have the following important property: If y is any point inside a**

minimal invariant set, J, there exists a c.t.o. λ such that the union of all transfinite iterate sets derived from f and y, equals J.

The minimal invariant sets thus constitute basic cyclic structures intrinsic to the behavior of all functions defined on an interval [-M,M] of R.