On Effectively Infinite Sets; or
Coping with Infinity in Modern Physics

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I. "Effectively Infinite Sets", an Introduction

In many of the less than rigorous arguments one finds scattered throughout the province of theoretical physics, a closely related group of them stands out for its ubiquity:

(1) The number N (of molecules, electrons, quanta, stars) is so large that one can treat it as "effectively infinite"

(2) The set X (notably in a gas) has so many molecules that one can replace causal calculations of trajectories (using Hamiltonians) by purely statistical calculations (incidentally bringing in other factors such as Entropy and a Second Law). Probabilistic and causal models may even be combined: thus the Equipartition Theorem depends on the present of "square terms" in the kinetic energy part of the Hamiltonian. This of course brings back Hamiltonians!

(3) Going in the converse direction: a remnant is considered so small

that one can treat it as "vanishing" or "negligible", or "effectively zero".

(4) One may also decide to treat particles as Euclidean points or "point Particles" . Their radius is so tiny that this impossible fiction makes no real difference. Let's (for example) proclaim that a certain potential (gravitational, Black Hole, electrical) is infinite as a single point! By pushing off this singularity onto a point, one can now treat everything else in its neighborhood as differentiable!

(5) If the scattering V of points on a Cartesian graph is bunched together closely enough, one might decide to treat it as an "effectively continuous". For applications of a different sort, one might prefer that it be treated as "dense", a very important distinction to mathematicians but one that may be lost on many physicists: a dense collection V will be totally disconnected, with arc length of measure zero. The closure of V however will be continuous.) The metaphorical image of "being bunched together" is itself based on the assumption of "negligible distances" between points.

In other words, for the number of points in an arc or segment to be Ef∞ the distances between "neighboring points" must be "effectively negligible". One then makes a decision, on the basis of the equations one prefers to work with (statistical, causal, continuous or differential), concerning whatever interpretation one chooses to adopt.

(6) The leap from "effectively continuous" to "effectively differentiable" may involve considerable difficulty. Isaac Newton was able to do so and give us his laws of motion. Likewise, 3 centuries later, Benoit Mandelbrot chose not to go that way,and demonstrated how useful the investigation of the fractal structures of discrete but self-similar Ef-∞ sets can be for the modeling of real phenomena.

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To determine that a set S is "effectively infinite" ("Ef-infinite" or Ef∞ or "ncc" (of an non- computable cardinality) , ***one must first enunciate the methodology Ω that is used for counting , weighing or otherwise "sensing" the cardinality of S.*** **Since so much of modern**

physics is absorbed in the macrocosm of the effectively infinite, or the microcosm of the effectively negligible, these considerations apply everywhere. However, no other branches of physics are more plagued with the dilemmas and difficulties of these distinctions as Thermodynamics, Statistical Mechanics, Quantum Statistics, the kinetic theory of gases, the study of "far from equilibrium systems" and related subjects . Notable in these fields are such far-reaching and permanently controversial notions such as the "Ergodic Hypothesis"; the "uniformity of molecular distributions" at equilibrium; the Maxwell-Boltzmann distribution of energies at constant temperature; the "Stoss-Zahl-Ansatz" (independence of molecular world-lines before collision (sometimes after collision!)); the Equipartition Theorem. Even the most basic and universally accepted concepts of Entropy and the Second Law depend upon sometimes dubious handling of "infinite" or " effectively infinite" quantities and magnitudes.

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DEFINITIONS

Definition 1 : **Given some counting procedure or methodology Ω , employed to ascertain the cardinal number of some set S with elements {e_j } , (***Note: the use of the letter j seems to indicate that the set has already been counted; here it is merely being used to indicate distinction.* **) we will say that S is "effectively infinite" (Ef-∞) or possesses an "ncc" (non-computable cardinal) if Ω is unable, through limitations of means or other problems, to determine the cardinal of S(=#S) . Effective infinity is linked, therefore to the** *methodology* **used making the count, and one could also write, S≡ ∞ (modulo Ω).**

Motivating Example: **One commonly finds the assertion in texts on Thermodynamics and Statistical Mechanics, that the sheer quantity of molecules in a "mol" of gas (6.06×10^{23}) forces one to abandon Hamiltonian mechanics and replace it by involking purely statistical methods.**

This view is not maintained consistently. Thus, Liouville's Theorem, (which states, essentially, that the phase space representation (PS1 or PS2; see further on) of a molecular system behaves over time like the flow of an incompressible fluid) is based on two assumptions which are quite different from the mere statement that a mol of gas is "effectively infinite"

(1) (a) The phase space is treated as much more than effectively infinite, but in fact as continuous;

(b) The Hamilton-Lagrange equations for this flow of a continuous fluid are used to demonstrate the invariant (phase-space) volume of this fluid.

(c) Hamiltonians return once again to bolster the almost mystical belief in the "equipartition of energy" every degree of freedom contributes energy in the amount $\varepsilon = kT/2$.

(2)In point of fact, however, one doesn't ever seem to use Liouville's Theorem to derive the basic equations of Thermodynamics! These can be derived by the classical techniques of Boltzmann and Gibbs, based on

dissecting phase-space into tiny boxes. Both the number of boxes, and the number of particles are "ncc". By once again assuming a continuous, indeed differentiable phase space volume, one can extract the Maxwell-Boltzmann distribution.

It is because the number of Hamiltonian equations for such a huge number of interacting systems raises the number of particles to ever more astronomical heights that one surrenders to the statistics of effectively infinite sets. The Second Law and the Entropy formulae will require the use of "effectively negligible" regions of phase-space, which we will come to in a moment.

Definition 2: **Given the methodology Ω , and ncc set S, S will be deemed "decomposable" if S can be divided into mutually disjoint sets $S= S_1 + S_2 + S_3 +\ldots S_N$, such that**

(a) S is ncc:

(b) Sj is cc for each index j.

(c) The number N is cc. This means that the cardinal of the set Q= {Sj} is itself cc modulo the methodology Ω .

Definition 3: **We will say that # S is *indecomposable* if:**

(a) S is ncc relative to Ω.

(b) Each subset Sj in the decomposition is cc

(c) The set of sets,Q= {Sj} will have an effectively infinite cardinal, no matter how the decomposition is made.

Definition 4:

(a) We will say that the *distance* d between two locations in a metric space, such as R^n, is *negligible* if d is below any of the methodologies available to compute the location or behavior of any system equal to or less than its length. An excellent example is the Planck length: we know what the number is, but can say nothing about any phenomenon at any smaller length.

(b)Let N_p be a neighborhood in R^n, centered on a location p. We will say that is negligible if its measure is below any methodology available that is able to compute its measure.

Definition 5: **We will say that a set of locations of a system S on a line is "effectively connected" or "effectively a segment" if it can be covered**

by an open covering of negligible neighborhoods, none of which is empty.

Definition 6: **Defining an "effectively continuous set" involves some subtleties. How does one distinguish between an "effectively dense set" and an "effectively continuous set"?** ***This problem may appear somewhat academic, yet it is a thorn in the side of the physical sciences*****. For example, a Bose-Einstein "cloud" is treated as "effectively dense", but the trajectory of a light quantum, which could be considered to be a series of "pulses" at a minimum frequency, is treated as "effectively continuous". In the Boltzmann expression for the Entropy, S= -k lnW, the phase-space "volume" is deemed to be effectively continous (to give meaning to the idea of a volume), but the particles and their collisions involved in Boltzmann's H-Theory, (or say, in Dirac's "hole hypothesis" for describing the behavior of positions in Quantum Field Theory) though unimaginably huge in number, are treated as effectively dense but discrete.**

Need one be reminded of the logical quagmire involved in the positing and employment of "point particles"? They are not "points" in any sense, they can be at most "effectively negligible neighborhoods" of points.

Definition 7: **Let S be an "effectively dense" set of "effectively infinite cardinality" (ncc) uniformly disbursed throughout R^3. We will say that S is** ***ergodic*** **if the closure of the intersection of any "effectively negligible neighborhood" with S is "effectively infinite".**

A definition of this sort is clearly needed as preparatory to the construction of such basic notions as Entropy and the Second Law of Thermodynamics. **Indeed one needs more: since Entropy is treated as a total differential in the defining equation dS = dQ/T, and since the basic mechanism for the increase in Entropy is inertial, that is to say, collisions or "effective collisions", the set of locations and momenta in a gas in equilibrium in a compact region of micro-canonical phase space, must be treated as** ***effectively differentiable*****.**

Now how does one define that? This is the best I can come up with on short notice:

Definition (8) **of an Effectively *Differentiable Set*: Let M be an actually differentiable surface of dimension < N, and class C^k, in a real Euclidean space R^N. Let A be a set of locations on M such that**

(1) The closure of A in M is a smooth sub-region of M

(2) A is ergodically dispersed, that is to say, one for which the intersections of the closures of all neighborhoods with A, even negligible ones, are effectively infinite.

We may then (cautiously) suggest that A is differentiable if, treating it as a sub-manifold of M, one derives useful and predictable results!

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The study of effectively infinite sets begins with the assumption that the determination that the cardinal number of a set, #S, is "infinite for all practical purposes" is linked to the nature of the Methodology Ω used in the measurement of some numerical quantity associated with S.

For example, one mol of a perfect gas has been determined to contain about 6.06×10^{23} molecules. It is obvious that this quantity is far too large be able to write out a table of Hamiltonian equations for each molecule and determine their collisions and trajectories. The methodology deemed inadequate is based on the classical dictum of Laplace, namely that if one knew all the positions and velocities of all the particles in the universe, one could completely calculate its entire past and future.

Putting "positions" and "velocities" (or momenta) on an equal footing moves the analysis from real (or configuration) space, to the more amorphous "phase space". From this point forward, everything takes place in the cosmos of thought experiments! Since the particulate content of phase space is inconceivably large, one can proceed in two directions:

(1): Discreteness is replaced by continuity. The considerations invoked at the beginning of this discussion now come into play: once again, assuming that the volume of phase space is compact, one cites both the forbiddingly huge number of particles, but also the assumption (and it is a hidden assumption which is at the heart of the circular

reasoning for the presence of uniform distributions at equilibrium), that the particles are all bunched so closely together, that one can treat them as a continuous fluid, and incompressible at that!

Continuing in this direction one invokes the great Theorem of Liouville, which tells us that this fluid is incompressible: the Hamiltonian flow is "phase space volume preserving". Note that one has brought back Hamiltonian dynamics into a supposedly purely statistical treatment!

If the flow is continuous and incompressible, it can only mean that *any simple neighborhood of any point contains an "effective infinity" of points:* **All open sets are ncc.**

In dealing with "huge sets" , the cardinal number of which are "permitted" to become infinite, or large numbers that are treated as infinities, (or some combination of the two) one encounters a sharp reversal of the normal functioning of roles. Sending finite quantities to the "infinite limit" implies that one is trying to fit a set of discrete data into some equation, dense, continuous or infinite. Thus it is the

"infinite" that serves as the *approximation to the finite*, a complete reversal of the normal situation in which one selects a "finite sample" from a huge or infinite collection (example, the points on the graph of a continuous arc) as an approximation to the full graph.

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One moves to the next level of "effective infinity" with the technique of partitioning Phase Space into minute boxes through "coarse graining". Staying with 6-dimensional microcanonical Phase Space, PS1 = $R^3(^x)M^3$ = Euclidean Space $(^x)$Momentum Space, the compact region occupied by the system is partitioned into a vast number of boxes, so huge in fact, that if the tiny boxes are indexed as B_1 , B_2 ,… , B_k,…*then the index set K={k} of this collection must, using our standard methodologies, also be treated as ncc: "effectively infinite".*

Each box is associated with a discrete energy, ε_k. Inside each box there is a collection of n_k molecules. When both the quantity

$$N = n_1 + n_2 + \ldots n_l$$

and the index set L = (1,2,……, l) are "sent to effective infinity", everything becomes _differentiable_, and one derives the Maxwell-Boltzmann distribution of energies. The Boltzmann Entropy Law drops quite naturally out of this questionable procedure, and one has the famous law S = klnW, W being the phase space volume.

We observe that although the Liouville volume is invariant, the Entropy, hence the volume W, is always increasing! These two conceptions of volume arise from the "continuity" arguments of procedure I, versus the discreteness, transformed into differentiability of procedure II!

Liouville's Measure-Preserving Theorem in Statistical Mechanics.

The applications of Liouville's Theorem in Statistical Mechanics are largely specious. When it is applied properly it is actually an argument _against_ the most basic of all its concepts: Entropy.

The first criticism comes from observing that Statistical Mechanics is by and large an _inertial_ or _kinematic_ theory, not a _dynamical_ one. Particles, molecules for example, may or may not collide, but such

collisions result only in an exchange of momentum. There are no force-fields and no potential energy. It follows that the "Hamiltonians" only have a kinetic energy term K, with no potential energy V. The kinetic energy is formed from the momentum coordinates p_k, and is constant before and after collisions. The potential energy would normally be formed from the coordinates qk, but these are absent.

Therefore the total derivative of the density flow is identically zero, except at the jump points (collisions) If and when it does change it does so by chaotic, non-differentiable leaps. But Liouville's Theorem belongs to the Calculus of Variations and depends upon the existence of variational derivatives.

Through statistical arguments that have nothing to do with Hamiltonians, it is argued that the equilibrium distributions in R^3 will be uniform: this is by far the most likely situation, and the actual equations of motion are irrelevant.

Parenthetically, if one moves to the macrocanonical phase space PS2 of N ~ 10^{23} dimensions, it is not entirely clear how one identifies a "uniform distribution" ***within*** **a single vector of 6N dimensions!**

One consequence of this must certainly be the all of the particles of some given ***energy*** **ε, will be uniformly distributed in momentum space, M^3 on spherical surfaces, whose radius is the square root of the energy times a constant of proportionality. Since the set of particles of a given energy ε is (apart from a negligible statistical deviation) deemed "effectively infinite", it must also be uniformly distributed throughout R^3 . What can we say about the volume of this uniform distribution of spherical surfaces. In some sense it should be taken as the the volume of the entire compact system S, which we can write as W_S.**

What about the distribution in Momentum Space? **Particles with the same energy ε can have differing values for the momentum in each direction; the energy functions as a metric. All of these particles must be on the same 3-sphere in M, with radius**

$R_\varepsilon = \sqrt{(2m)\varepsilon} = \sqrt{(mv_x)^2 + (mv_y)^2 + (mv_z)^2}$

Given that the distribution in R^3 is chaotic (or ergodic, etc.) one has an effectively infinite set of "spherical rings" in 6-dimensional micro-canonical Phase Space. No doubt one could perhaps make some sense out of the notion that this quantity is preserved, but one would need more than Liouville's Theorem to do so.

This picture also seems to cast into obloquy the notion that one can partition Phase Space into little boxes, each of which has a volume proportional to the energy of the particles contained in it. For the distribution of the particles in R^3 is uniform, the distribution of the momenta on an "energy sphere" constructed above is also uniform, and there is no way, to my mind, of taking all these "spherical cylinders", and *rearranging them into a 6N dimensional box* "proportional " to the energy.

The Maxwell-Boltzmann distribution in fact does not need Liouville's Equation, or the decomposition by boxes, but can be derived

directly from statistical arguments dating back to Jakob Bernoulli in the 18th century.

A proper application of Liouville's Theorem it will lead to Birkhoff's and Poincare's recurrence Theorems in Ergodic Theory which show that a measure-preserving transformation on a compact space comes infinitesimally close to its initial conditions infinitely often. This completely undermines the notion of Entropy and was the basic of the objections of Zermelo and Loschmidt against Statistical Mechanics.

Let T be the trajectory of a system S that represents the state of a quantity of gas, G, with an effectively infinite number N of molecules, (say to the order of $N= 10^{23}$). By inventing the "macro-canonical ensemble" in 6N dimensions, Gibbs ingeniously solved the problems involved in misrepresenting this as a continuous flow of a huge number of discrete numbers in symplectic 6-dimensional space. Rather, he simply represents each molecule as 6 of the dimensions of an R^{6N} space, reducing the entire system to a point, and the trajectory in time to an arc. One no longer has to pretend that one's "fluid" is a 6-dimensional

Euclidean volume, one can instead create a *genuine* volume in 6N space by looking at the set of all possible initial conditions compatible with the constraints on the system. One can take actual derivatives and integrals and invoke Liouville's Theorem which, in this form, asserts the conservation of the phase space distribution and the density along any trajectory T.

To our mind, only part of this result is correct, namely, that the equation of continuity is zero. However, for Liouville's Theorem to apply, one also needs the vanishing of the variation of the Hamiltonian (*The Wikipedia article on Liouville's Theorem explains this very well.*) and the fact that the velocity field (dp/dt,dq/dt) has zero divergence. For simplicity, one can assume that all of the molecules have identical mass, that their size is not quite, but essentially negligible, and that their motions are completely uncorrelated, (Stoss-Zahl-Ansatz, at least before collision!)

One would like to eliminate collisions as well, but unfortunately this cannot be done *as this leaves no mechanism for the evolution of Entropy* *and the action of the Second Law. Without these things there is no science of Thermodynamics.*

Therefore, when drawing up the long list of coordinates that go into the macrocanonical representation of a system (A single point in R^{6N} space) when listing the coordinates of the trajectory T, one must take into account the collisions between molecules. These produce abrupt discontinuities in the momentum coordinates, and singularities in the derivatives of the local coordinates.

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Let us recall that in Statistical Mechanics, Phase Space comes in two varieties:

(A): PS1. This is a 6-dimensional Euclidean/Symplectic space filled with all the particles of a system S, each of which is represented by 6 numbers, 3 Cartesian coordinates (x,y,z) and the corresponding 3 momenta (mv_x, mv_y, mv_z). Normally all the particles are identical bits of substance (*the old quarrel about the "absolute identity" of electrons*), and the mass, m, can be omitted. Then, apart from a constant of

proportionality, the energy is simply the square of the Euclidean metric of the momentum space E =1/2(v_x^2+ v_y^2+ v_z^2)

(B): PS2. This is a Euclidean space of 6N dimensions, where N is in the neighborhood of 1023. There are 6dimensions for each particle, and an entire system is pictured as a single point moving along its "world-line", propelled by a combination of initial conditions and a Hamiltonian equation combining kinetic and potential energy.

This is the important point: the system S in PS1 is "effectively infinite" in the sense of this article. One therefore abandons causal for probabilistic descriptions of the global behavior of S.

One does not need the notion of an effectively infinite set for the representation by PS2, since this vector space is simply a Euclidean continuum of a gigantic number of dimensions. However, the distinction between "actual infinity" and "effective infinity" returns in a new guise.

In the application of Liouville's Theorem one integrates over the transverse topology (one sees here the origin of certain models of Non-Commutative Geometry), where the region of integration consists of all

world-lines with all possible initial conditions at some point of origin in time, t =0. One therefore constructs a tiny sphere of 6N-1 dimensions around some point p on the trajectory T, and argues that, by Liouville's Theorem, the volume of this sphere (which, for a small radius must surely be infinitesimal, and for a large one effectively infinite!) is invariant along its whole trajectory.

There are serious problems with this model, starting with the fact that every such sphere drawn around a point p in PS2, and intersecting all local contiguous world-lines, must be densely riddled with collision points, that is to say, jump discontinuities in the momenta, and singularities in the time derivatives of the motions.

The argument is very simple: if the Second Law applies universally, and if Temperature and Entropy are uniformly distributed in every sub-region of phase space occupied by a system, then it they must also be present in *any sphere around a point in PS2*.

Now the only mechanism for the maintenance of Entropy and the Second Law, is the collision mechanism. In equilibrium there are no

external forces fields, what motion there is generated by heat, and heat is generated by collisions. Since heat is assumed to be equitably distributed in the system S, the density of collisions must be uniform throughout the representation of the system in PS2.

This being the case, it is absurd to speak about differentiation or integration in a plenum riddled with singularities everywhere! **Differential equations are inconceivable, Hamiltonians make no sense, and there is no volume integral to justify the application of Liouville's Theorem.**

Furthermore, imagine that two molecules are headed on a collision course in a pair of world-lines T_1, and also T_2, both in PS2. By an infinitesimal perturbation of say, T_1, this collision can be averted: The system in T_1' will pass by the collision location present in T_2. Thus, even the pattern of collisions is unstable; trajectories very close to each other may represent wildly different behaviors!

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Conclusions

All measurement that is more detailed than simply "larger and smaller" involves counting. When an object is weighed one r ecords the number of pounds. There are situations in which one places an object against a set of qualities, such as colors, for purposes of comparison, but our concerns have to do with the role of "effectively infinite" and "effectively negligible" sets in quantitative science.

Physics is concerned with the revelation of the laws that govern the world around us. This "phenomenal" is apprehended by us in sets of data, finite collections of numbers. When mathematics is applied to these sets to derived models, hypotheses and laws, one has to bring in infinities.

At the same time, mathematicians have enormously increased our understanding of the richness and complexity of the concept of infinity since George Cantor's insightful work in the 1870's. We can no longer pretend that we live in a relatively simple Galilean world (who was

content to remark that the set of integers, and the subset of their squares have the same cardinality).

Therefore, when a physicist is dealing with an exceptionally large potential dataset that needs to be treated as effectively infinite, he/she must make a choice between dense models, self-similar fractal models, continuous fluids, incompressible flows, differentiable surfaces, sets of countable cardinality (integers and rationalities), uncountable cardinality (the real line and R^n spaces), or even second order uncountable cardinality (function spaces and operator algebras)!

Nor can topology be neglected in the clarification of the notion of an effectively infinite set: a "neighborhood" can be modeled as an infinitesimal set, an open set, the germ of a set of functions, the basis for a sheaf or a manifold, a minimal coordinate patch, etc…

The choice of which model one chooses to use for effectively infinite sets, can be a difficult one, and often requires a theoretician to be as

much artist as scientist. This was certainly true in the case of Ludwig Boltzmann, and is why we continue to honor him to this day.