Report on the
Bolyai-Gauss-Lobachevski conference on
Non-Euclidean Geometry and its Applications to Physics
Bessenyei György Tanárképzö Föiskola
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Istvan Lovas, one of the two organizers, (the other was Lazlo Jenkovsky) opened the conference by defining its purpose: to bring together scientific communities around the world that had lost contact during the Cold War. It was this consideration that had determined the choice of subject: the 3 mathematicians credited with the discovery of non-Euclidean Geometry , Janos Bolyai, Carl Friedrich Gauss and Nicolai Ivanovitch Lobachevsky, lived in 3 countries seriously affected by the Cold War. Accordingly most of the delegates were from Russia, the Ukraine, Hungary and Romania, (which now includes the part of Hungary where Bolyai did his work) .

Significantly , although Gauss was a German, there were no German participants. Beyond the rifts caused by the Cold War there are, of course, the more serious ones caused by WWII .

After these the largest contingent came from Italy. Italian mathematics has always been strong in geometry. The cultural exchange between Italy and Romania, disrupted in the 40's, dates to the Renaissance. Remaining delegates were from Poland, Norway, England, the United States, Japan, Egypt, and France.

Both historical and technical talks were presented. The lectures included topics from the fields of non-Euclidean geometry, projective geometry, foundations of mathematics,

Hilbert Space, classical mechanics, general and special relativity, elementary particle theory, quantum gravity, and cosmology.

Although the official language of the conference was English, many of the speakers were not conversant in it. For the first day of the conference and part of the second, Jenkovsky stood in the arena at the front of the auditorium's descending semi-circles of rows of seats and performed consecutive translations into English from Russian, Ukrainian, Hungarian and Romanian! He also speaks Italian, German and French.

Historical Papers

Janos Bolyai was the subject of several talks on the first day. Bolyai (1802-1860), an officer in the Austrian army, was the son of Wolfgang Farkas Bolyai, himself a professional mathematician. In fact, Bolyai's ideas on hyperbolic geometry (one of the 3 non-Euclidean geometries) were first published as an appendix to his father's standard geometry textbook (*Tentamen* , 1831) .

2002 is the Bolyai Centennial. Owing to a freer environment in international scholarship most of Bolyai's mathematical research , deposited in the university at Cluj, Romania, is being examined, edited and published for the first time . Bolyai left 14,000 pages of written manuscript, much of it about mathematics, with many original results in number theory as well as geometry. Only a small portion of this vast archive has been examined; he may have done research in other fields as well.

Bolyai's researches into Euclidean Rings preceded those of Carl Friedrich Gauss by a number of years. He appears to be the first mathematician to have investigated the algebraic structure of what are now called   "Gaussian Integers", numbers of the form z

= x+iy, where x and y are integers and i = √-1 . His papers contain proofs that

    (i) the ring Z[i] , admits unique factorization

    (ii) that all primes of the form p = 4m+3 are also primes in Z[i]

    (iii)   that all primes of the form p = 4n+1 can be written as the sum of two squares , p = a^2 + b^2 , which can be factored as

    p = (a+ib) (a-ib) . Bolyai's work dates from 1825; Gauss would not pick up this subject for another 10 years. (Priority of publication is crucial in mathematics, being a subject in which anything can be discovered by anybody at any time!)

    During one of the presentations on Bolyai a sonnet written by the Hungarian poet, Mihaly Babits, in praise of non-Euclidean geometry was recited, first in Hungarian, then in Russian, Romanian, English and French. The French translation was delivered by the lone French delegate Christian Velprey. He read it very well, then commented that he disagreed with the sentiment expressed by the poem. At which point someone cried out : "You can't disagree! It's a poem!" The idea with which Velprey disagreed appears in the last two lines, the "epigram" of the sonnet. It states that Bolyai freed mankind from blind slavery to Euclid. "Euclid didn't enslave anyone!" Velprey exclaimed.

    There was only one paper, a very good one, devoted to Nicolai Ivanovitch Lobachevsky. It was delivered by N.A. Chernikoff, a famous Russian mathematical historian . In 1826 Lobachevsky sent his researches to Ostrogadskii, (*the second half of the Stokes-Ostrogadskii Theorem*) . Ostrogadskii thought he was a crank and sent them back. But in 1829 Lobachevsky was

promoted to rector of Khazan University and could publish anything he wanted. *The Russian academy then banned the teaching of his ideas* . One had to go to Germany to study them.

Likewise, Janos Bolyai's work was censored by the Hungarian Academy of Sciences. Carl Friedrich Gauss practised self-censorship. Although the most powerful figure in European mathematics, he not only refused to publish his findings in non-Euclidean Geometry, he didn't permit any of his graduate students to do research on that subject!

The tacit though total proscription of non-Euclidean Geometry by the European academic world suggests the strong impress of Hegelianism, based in its turn on the philosophy of Immanuel Kant. Kant argues in the *Critique of Pure Reason* that Euclidean Geometry is a *synthetic apriori* , an innate idea which organizes experience, without which it is impossible to think about the world . His exact words are " *The concept of Euclidean geometry is by no means of empirical origin, but is an inevitable necessity of thought."* In the global sense, Kant was clearly wrong. However the 3-standard non-Euclidean geometries revert to Euclidean geometry in the limit of the infinitely small.

Neither Bolyai, Gauss nor Lobachevsky received recognition for their discoveries in their lifetimes. It was only in the 1870's that people started taking non-Euclidean geometry seriously. And it was only in 1900 that David Hilbert acknowledged their role in its discovery. So much for those people who believe that mathematicians are devoted to "pure inquiry"!

Non-Euclidean Geometry and Relativity

Several interesting talks related the early work in non-Euclidean Geometry to developments in Special and General

Relativity. Bolyai, Lobachevsky and Gauss asked the question , which of the 3 standard non-Euclidean geometries best describe the actual universe? Gauss even conducted a famous experiment in which lanterns were flashed from 3 peaks in the Alps , to see if the "angle deficit" corresponded to flat, elliptic, or hyperbolic geometry. The apparati were too crude to detect any deviation from flatness. Even today, General Relativity experiments involve extremely minute quantities.

T. Weszeley read a paper entitled :"*Which geometry do we live in?*   The 3 kinds of evidence that are normally considered are:

> (a) Anomalies of the planetary orbits.
>
> (b) Modifications of Newton's Law of Gravitation
>
> (c) Bending of light in the presence of matter.

(a)

These were already being observed two centuries ago when it was noticed that the advance of the perihelion of Mercury, (the furthest orbital point from the Sun), was off by an amount of 43 seconds of arc per century. Einstein's explanation involves a hyperbolic space-*time* . This replaces the universal homogeneous metric of non-Euclidean geometry by the differential forms of Riemannian geometry.

(b)

Under the assumption of a hyperbolic universe in which Kepler's Law : "Planets sweep out equal areas in equal times" pertains, Newton's Law of Gravitation is modified as follows:

$$F = -\frac{GmM}{k^2 \sinh^2(r/k)}$$

r is the radius vector from the sun to the planet, k is minus the hyperbolic curvature. When r/k << 1 one gets:

$$F \approx -\frac{GmM}{r^2} + \frac{GmM}{3k^2}$$

The extra term on the right is something that can be measured. It appears in the correction for the perihelion to Mercury predicted by General Relativity. It generalizes to a form of the deSitter metric. deSitter spaces have the property of an absolute curvature, even in the absence of matter.

Beltrami put the metric into the form:

$$d\sigma^2 = (k^2 dr^2 / r^2) + r^2 d\varphi^2$$

Observe that letting k ---> ∞ one does *not* obtain the Euclidean metric, although k = ∞ is the condition for Euclidean space. One sees that deSitter's solution of the Einstein equations is already present in the writings of Bolyai and Lobachevsky in the 1830's

(c)

A bending of less than 1 second of arc was detected during a total eclipse of the sun by the Arthur Eddington expedition to Principe Island off West Africa in 1919. Incredibly, (and somewhat suspiciously) a single observation was accepted as confirming evidence of a theory describing the behavior of everything in the universe!

Even after accepting the data, 2 interpretations are possible:

(1) The universal geometry is non-Euclidean

(2) Geometry is Euclidean, but light doesn't move in straight lines.

Although the first interpretation has won universal acceptance, it is still not forbidden to play around with the

second. There are 2 reasons why the first interpretation has prevailed:

(1) The simplicity of a model in which the trajectory of a ray of light can be described by refraction in a non-homogeneous medium; and

(2) The desire to rescue the idea of a homogenous space in the absence of matter, that is to say, empty space or the "void".

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Conics in non-Euclidean Spaces

A talk on this subject was given by Christian Velprey, of the mathematics research faculty at Jussieu, Paris. He studies the properties of conic sections in non-Euclidean geometries. It is natural to view the 3 geometries as aspects of a single "Pan-Geometry", particularized by inequalities:

Let S, C, T, G stand for, respectively sine , cosine, tangent , cotangent, in either Euclidean , spherical, or hyperbolic 2- space. Then these relations are the same for all 3 geometries:

(a) $T = S/C$

(b) $G = T^{-1}$

(c) $C^2 + \gamma S^2 = 1$

The difference between them is in the value of γ , $= 1$, <1 , or > 1

Kepler's and Newton's Laws in hyperbolic and elliptic space give rise to families of conics which one doesn't find in Euclidean space.

Kepler's Laws state that:

(1) All planets move around the sun in elliptical orbits

(2) They sweep out equal areas in equal time.

(3) The squares of the periods of the planetary orbiting times are proportional to the cubes of their mean distance from the sun.

The *barycentric theorem* of Newton is no longer applicable in elliptical geometry. This theorem is basic to statics and classical mechanics: it states that the dynamics of any material system in a compact region of space can be decomposed into the motion of the center of gravity, and the motion around it. In particular one can show that the inverse square law allows one to treat the gravitational effect of a massive object on the rest of the universe as if that object were concentrated at its center of gravity. The theorem is very sensitive to the exponent. Had the law of gravitation included an exponent for r in the amount $-(2 + \varepsilon)$ it would have been detected long ago.

In elliptic geometries interior and exterior are not unambiguously defined. To calculate the momentum means assigning a direction of motion. Calculating the total field strength between two points depends on the direction in which the field is calculated. Two masses on a circle attract around both segments.
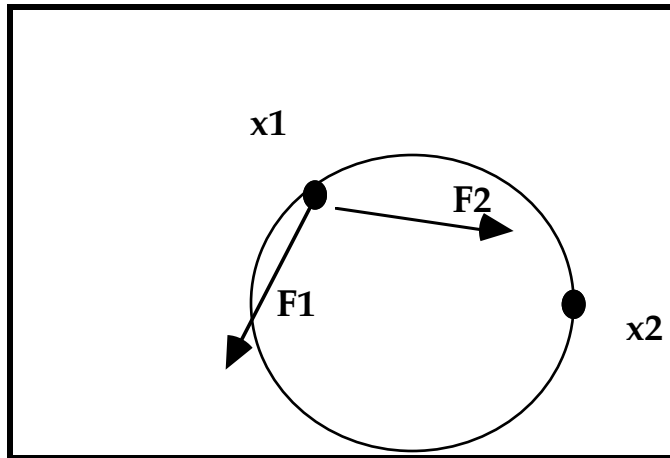
$$F_1 = -\frac{\gamma mM}{r^2}$$

$$F_2 = +\frac{\gamma mM}{(R-r)^2}$$

$$\hat{F} = \gamma mM\left(\frac{1}{r^2} - \frac{1}{(R-r)^2}\right)$$

In more than one dimension one must integrate over all geodesic lines connecting the two bodies.

Technical Lectures

These were difficult to follow and are even more difficult to relate to a non-technical audience. There were some exceptions, among them the talk by Lazlo Jenkovsky: *Regge trajectories and non-Euclidean Geometry.*

The subject is relevant to high energy strong interactions of elementary particles such as quarks and gluons in the relativistic regime. Many questions persist about the hypothesis of *quark confinement* : How does it function ? Is it indispensable? Are there geometric structures or mathematical schemes that can adequately represent it?

Quark/gluon theory is essentially a generalization of the exchange relations of meson reactions. The quantum numbers , normally treated as discrete quantities, undergo continuous modifications

Regge trajectories, generalizations from the classic harmonic oscillator, allows one to conceptualize a 'breaking string' model for quark confinement. Jenkovszky showed the connection between these to the "q-deformations" of the currently fashionable theory of quantum groups. However, the commutator relations for the interactions under discussion were written down by physicists long before the abstract theory of quantum groups emerged. Jenkovszky also suggested that there might be a non-commutative geometry based on these q-deformed matrices.

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