The General Collision Theorem
Proof and Consequences
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The proof of the General Collision Theorem depends in a fundamental way on the Restricted Collision Theorem:

Restricted Collision Theorem:

Let S be a system of massive particles in a 1-dimensional universe with inertia but no gravitation.

$$\boxed{S = \{p_1, p_2, \ldots. p_N\}}$$

Each (point) particle p_i has mass m_i .The initial conditions are assumed given: $\boxed{p_i = (m_i, a_i, v_i), i = 1, \ldots, N}$ ***, at time t =0., N being a finite number. By fixing the origin and rest frame of the system at the center of gravity, one has the 3 conservation laws:***

Conservation of Moment

$$\boxed{\sum_{i=1}^{N} m_i x_i = 0}$$

Conservation of Momentum

$$\boxed{\sum_{i=1}^{N} m_i v_i = 0}$$

Conservation of Energy

$$\boxed{\sum_{i=1}^{N} m_i v_i^2 = 2E = const.}$$

Under these conditions, the total number of collisions of the system, both before and after t, is finite.

We assume that the Restricted Collision Theorem has been proven in all that follows. This being the case, all such systems will have associated with them a *collision number* c, an integer which is a function of the initial conditions.

General Collision Theorem

Keep the initial masses and velocities fixed , but allow the initial positions ai to vary freely, (under the constraint of keeping the rest frame origin at the center of gravity).

Then the total number of all collision numbers for all possible distributions of the a_i , will be finite.

The proof of the Restricted Theorem is simple and straightforward. The proof of the General Theorem appears to be unavoidably difficult; at least until a simpler version presents itself.

Preliminaries

All systems can be put into a simple canonical form:

Since the total number of collisions is finite, one can assume that the initial state of the system is located sometime t(=0) *before* any collisions have taken place. A bit of thought will show that S assumes the following simple form:

The sequence of initial velocities { v_i } separates into two sets, a set of positive velocities $\boxed{v_1 \geq v_2 \geq \ldots \geq v_s}$ and a set of negative velocities $\boxed{|v_N| \geq |v_N| \geq \ldots \geq v_{s+1}}$ to the right of these. ("Speed" ois always positive; "velocity" may be positive or negative) Note that the divide can occur anywhere, for it is only the locations that

figure in the equation for the Moment that determines the origin. However, because of the conservation of Momentum one has:

$$\boxed{\begin{aligned}&m_1v_1+\ldots+m_sv_s = m_{s+1}(-v_{s+1})+\ldots+m_N(-v_N)\\&= m_{s+1}|v_{s+1}|+\ldots+m_N|v_N|\end{aligned}}$$

(That these are the correct forms for the inequalities can be seen through playing the system in reverse time and noting that if, for example , v_2 were larger than v_1, a collision would have occured in the past.).

Another consequence of this distribution of velocities is that every particle must collide at least once with its immediate neighbors:

Lemma **: A collision between particles pk and pk+1 must occur if either:**

(1) v_k is positive and v_{k+1} is negative; or

(2) v_k and v_{k+1} are both positive with $v_k > v_{k+1}$; or

(3) v_k and v_{k+1} are both negative with $|v_{k+1}| > |v_k|$

Proof: **If two particles are headed towards each other, whatever impinges on them before the collision must increase their speeds. This establishes (1)**

If two particles are moving in the same direction in such a way that, if isolated, one would collide with the other, then whatever impinges on the faster particle will increase its speed , while whatever collides with the slower particle will either slow it down or reverse its direction, thus increasing the speed at which it moves towards collision. This establishes (2) and (3).

Collision Matrices and Collision Number

Consider first a two-particle system S = {p_1, p_2 }. Both rest frame and origin are chosen arbitrarily, so that the conditions at time

t= 0 are $\boxed{p_1 = \{m_1, a_1, v_1\}; p_2 = \{m_2, a_2, v_2\}; t = 0}$

The conservation laws are:

$$\boxed{\begin{aligned} &m_1 v_1 + m_2 v_2 = const. = m_1 v_1' + m_2 v_2' \\ &m_1 v_1^2 + m_2 v_2^2 = 2E = const. = m_1 (v_1')^2 + m_2 (v_2')^2 \end{aligned}}$$

where v_1' and v_2 ' are the velocities after collision. The equations of motion before collision are:

$$\boxed{x_1 = v_1 t + a_1; x_2 = v_2 t + a_2}$$

Given these initial equations one can show through straightforward calculation that the velocities after collision are related to those before collision by the matrix formula:

$$\boxed{\begin{aligned} &\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &A = \begin{pmatrix} \dfrac{m_1 - m_2}{m_1 + m_2} & \dfrac{2m_2}{m_1 + m_2} \\ \dfrac{2m_1}{m_1 + m_2} & -\dfrac{m_1 - m_2}{m_1 + m_2} \end{pmatrix} \end{aligned}}$$

Note that neither the a's nor the v's occur in the matrix: A is always the same for every collision.[1] This is quickly generalized to the following

Theorem : **If particles p_k and p_{k+1} collide, then the velocities after the collision are related to those before the collision by the formula:**

$$\boxed{\begin{array}{l} \begin{pmatrix} v'_k \\ v'_{k+1} \end{pmatrix} = C^k \begin{pmatrix} v_k \\ v_{k+1} \end{pmatrix} \\ \\ C^k = \begin{pmatrix} \dfrac{m_k - m_{k+1}}{m_k + m_{k+1}} & \dfrac{2m_{k+1}}{m_k + m_{k+1}} \\ \dfrac{2m_k}{m_k + m_{k+1}} & -\dfrac{m_k - m_{k+1}}{m_k + m_{k+1}} \end{pmatrix} \end{array}}$$

For describing the history H of a system in terms of its collisions and the order in which they occur, C^k can be incorporated into an $N^x N$ matrix A^k as a 2x2 diagonal element.

$$\boxed{A^k = \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & . & & & 0 \\ . & & C^k & & . \\ . & & & . & . \\ 0 & . & . & 0 & 1 \end{pmatrix}}$$

with N-2 1's on the diagonal, C_k filling up places (k,k+1)...(k+1,k+1), and 0's everywhere else.

The Generalized Collision Theorem can now be restated as :

Given initial conditions of mass, velocity for n particles on a line, the number of collision numbers attained by arbitrarily assigning locations $x_1 < x_2 < x_3 .... < x_N$. is finite.

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[1](**Note that the eigenvalues for such collision matrices are always i and -i, a result which has some bearing in symplectic or Hamiltonian manifolds.)**

Corollary: There is a maximum collision number.

Obviously there is always a minimum collision number. In fact, if all the velocities are distinct, it is always 1! One places the particles relative to the Center of Gravity in such a way that they all collide simultaneously at the origin. The situation in which 2 or more particles have an identical velocity is a limiting case in which the particles of equal velocity are placed together on the same location.

Equivalent Histories

There are several ways to describe the "history" of a system $S(a_i ; v_i ; m_i)$ $i = 1,2,\ldots N$, where the initial conditions are assumed to apply before any collisions have taken place. A complete description of the "history" of S, from first to last collision can then be written as:

$$\boxed{v^e = H(v^0) = A^{j_c} A^{j_{c-1}} A^{j_{c-2}} \ldots A^{j_1}(v^0)}$$

where the indices j_i are chosen among the indices of particles p_1 $\ldots p_N$, and

$$\boxed{\begin{array}{l} v^0 = \begin{pmatrix} v_1^0 \\ \cdot \\ \cdot \\ v_N^0 \end{pmatrix} \\ \\ v^e = \begin{pmatrix} v_1^e \\ \cdot \\ \cdot \\ v_N^e \end{pmatrix} \end{array}}$$

are the initial and terminal velocity values respectively.

If H is presented in its numerical form one will not be able to reconstruct the pattern of collisions, or even the individual collisions which transform the initial velocity vector into the terminal one. To do this we have to write down

(i) The sequential product of collision matrices in temporal order. That is, if the j-th collision occured before the j+1-st collision, the order of collision matrices is $\boxed{\ldots A^{k_j} A^{k_{j+1}} \ldots}$**. If the collisions are simultaneous, then the order is immaterial.**

(ii) The numerical entries in all the A^{kj} are replaced by ***formal indeterminates*** **, or mass letters. For example, if the entries for A^1 are c_1 ,c_2, c_3, c_4, then for these numbers one makes the following substitutions:**

$$\boxed{\begin{pmatrix} c_1 = \dfrac{m_1 - m_2}{m_1 + m_2} & c_2 = \dfrac{2m_2}{m_1 + m_2} \\ c_3 = \dfrac{2m_1}{m_1 + m_2} & c_4 = -\dfrac{m_1 - m_2}{m_1 + m_2} \end{pmatrix}}$$

One then replaces the numerical entries in H ***by all the rational functions in the masses m_1 ...m_N derived from the formal products of the terms in the collision and multiple collision matrices.***

Although the sequential changes in the entries in the velocity N-vector after collisions ***cannot be derived from the formal matrices*** **, the formal matrices can all be derived from the velocities and of the (numerical) configuration of velocities and positions of the system after collisions. What this is saying that the indices of the particles involved in the next collision are directly derivable from the velocities and positions of the particles themselves.**

We therefore define

(1) The *material history* of an N- system $S(a_i ; v_i ; m_i)$ consists of all the collision matrices with their numerical entries arranged sequentially in the temporal order in which collisions occur, as well as the list of all the velocity N-vectors and position N-vectors that are the result of such collisions. From these vectors, the index of the next collision or multiple collision matrix can be unambiguously determined.

(2) The *formal history* of an N- system $S(a_i ; v_i ; m_i)$ consists of all the collision matrices with their numerical entries replaced by expressions in the indeterminate mass letters, and the entries in H replaced by the formal rational functions obtained through matrix multiplication. One cannot derive the sequence of collisions from the formal history, which depends upon the data furnished by the material history .

We are almost ready to define equivalence between systems. Basically, two systems with identical initial velocities and masses but differing initial locations are equivalence if there is a continuous deformation of one into the other that preserves the formal history up to the transposition of commuting matrices. One does not need to maintain strict temporal order if the global collision history is unaltered.

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What happens if there is a multiple collision? For example, if 3 particles k, k+1, k+2 collide at the same place and time, then the matrix of this collision cannot be either the product $A^k A^{k+1}$ or $A^{k+1} A^k$, since these are not formally equal. Instead the collision matrices are replaced by a *multiple collision matrix* M^k , where k

is the left-most particle in the triple collision. We will return to this in the following section.

Imagine first a system of 3-particles p_1, p_2, p_3 . Placing the origin of the rest frame at the barycenter, the initial velocities before a triple collision will be transformed by the collision to:

$$\boxed{v_1 \to -v_1;\ v_2 \to -v_2;\ v_3 \to -v_3}$$

In this very special situation the mass numbers (or letters) are irrelevant. Suppose next that the particles are in a system S, indexed as p_k , p_{k+1}, p_{k+2} , with velocities v_i *relative to the barycenter* of the *entire* system . One then computes the velocity u, of the barycenter of this 3-particle system, which by the conservation of Momentum is:

$$\boxed{(m_k + m_{k+1} + m_{k+2})u = m_k v_k + m_{k+1}v_{k+1} + m_{k+2}v_{k+2}}$$

The calculation of the transformation of velocities by this triple collision is done by subtracting u and applying the reflection of velocities cited above. The result may be summarized as follows:

The collision matrix $M^k_{(3)}$ for a triple collision of particles p_k , p_{k+1}, p_{k+2} is given by :

$$\boxed{\begin{aligned} &M^k_{(3)} = \\ &\frac{1}{m_k + m_{k+1} + m_{k+2}}\begin{pmatrix} m_k - m_{k+1} - m_{k+2} & 2m_{k+1} & 2m_{k+2} \\ 2m_k & m_{k+1} - m_k - m_{k+2} & 2m_{k+2} \\ 2m_k & 2m_{k+1} & m_{k+2} - m_k - m_{k+1} \end{pmatrix}\end{aligned}}$$

As simultaneous collisions of 4 or more particles are immediate generalizations of this construction, there is no need to go into them here.

Let s,t be real numbers. If $s < t$, then the interval between and including them is given by $I = [s,t]$.

Definition: **$|[s,t]|$ shall represent the interval *between* s and t . If $s \leq t$, then $|[s,t]| = [s,t]$. If $s > t$, then $|[s,t]| = [t,s]$. Obviously $|[s,s]| = \{s\}$. Equally clear is the expression** $\boxed{z \in |[s,t]|}$

Let the two systems

$$S_a = S(a_i;\ v_i;\ m_i) \text{ and } S_b = S(b_i;\ v_i;\ m_i)$$

$i = 1,2...N$ be given, with identical masses and initial velocities, but arbitrary initial locations $\{a_i\}$ and $\{b_i\}$, subject only to $a_i < a_{i+1}$, $b_i < b_{i+1}$, $i = 1,2,3,...N-1$

Definition : **S_a and S_b are *equivalent* if**

(1) The formal history of S_a is identical to the formal history of S_b ;

(2) The decomposition by collision and multiple collision matrices is the same, perhaps permuted in their temporal order

(3) Let E represent the "box" defined by

$$\boxed{\begin{array}{l} e \in E \rightarrow e = (e_1, \ldots e_N)\ \& \\ e_i \in |[a_i, b_i]|, i = 1, 2, \ldots, N \end{array}}$$

Then the system $S_e = S(e_i;\ v_i;\ m_i)$ has the same formal history and decomposition by collision and multiple collision matrices for all $e \in E$.

Clearly, equivalent systems have the same collision number. The General Collision Theorem states that the number of equivalence classes is finite.

The Standard Position Theorems

Definition: **Let S be a system with all initial parameters specified. Let us say that a multiple collision occurs at the jth place in the sequence of collisions. The N-velocity vector after that collision has entries $\boxed{v_1^*,\ldots\ldots,v_N^*}$ which are linear functions of the initial entries $\boxed{v_1^0,\ldots\ldots,v_N^0}$, with coefficients that are rational functions of the masses of these entries. Not all of the particles in the system need enter into the linear expressions for the values of the after-collision velocities.**

We say that a particle p_k is *responsible* **for a multiple collision if its initial velocity enter into the linear expressions for the particle velocities after that collision. The collision of p_k with an adjacent particle, once or many times, preceded and was responsible for causing the collision**

First Standard Position Theorem: **Let c_{n1} ,,, c_{nq} be all the multiple collisions for which some particle p_k is responsible. Then there is a neighborhood around a_k , (the initial position of p_k) in which any point $a_k' \neq$ ak will** *not* **be responsible for** *any* **multiple collisions in the modified system**

$$\boxed{S' = S(a_1,\ldots a_{k-1},a_k',a_{k+1},\ldots a_N)}$$

We sketch the proof for triple-collisions, from which the generalization to n-fold collisions is immediate.

A simple collision of particles i and i+1 is of the form:

$$\boxed{\begin{aligned} x_i &= v_i t + a_i \\ x_{i+1} &= v_{i+1} t + a_{i+1} \end{aligned}}$$

The collision location is given by:

$$\boxed{\hat{x} = \frac{v_{i+1}a_i - v_i a_{i+1}}{v_{i+1} - v_i}}$$

This is linear and homogeneous in the initial locations, with coefficients determined by the initial velocities and the collision matrices. After a certain number of collisions have taken place, the locations assume the form:

$$\boxed{\begin{aligned} x_j &= c_1^j a_1 + \ldots\ldots c_N^j a_N \\ x_{j+1} &= c_1^{j+1} a_1 + \ldots\ldots c_N^{j+1} a_N \\ x_{j+2} &= c_1^{j+2} a_1 + \ldots\ldots c_N^{j+2} a_N \end{aligned}}$$

Now a triple collision (*relative to the barycenter of this 3 particle system*) occurs when the following set of ratios are equal:

$$\boxed{\frac{x_j}{v_j^*} = \frac{x_{j+1}}{v_{j+1}^*} = \frac{x_{j+2}}{v_{j+2}^*}}$$

these being the velocities attained at the moment just before this collision. It is clear that a tiny perturbation of *any single initial position a_k* , (all others being left fixed) , will alter these ratios so that no triple collision occurs. Furthermore, if this is the *first* triple collision for which a_k is responsible, the alteration can be made so small that the expressions for the new velocities will not be changed by that alteration .

Having "uncoupled" the first multiple collision, one precedes to uncouple the next one by reducing the size of the perturbation. As the total number of multiple collisions for which a_k is responsible must be finite, one takes the intersection of all these perturbations to obtain a neighborhood around a_k , such that every real number in that neighborhood except ak itself will not

be responsible for any multiple collisions in the modified system of initial conditions.

Corollary: Given a system S with initial locations $\{a_i\}$, there exists a hyperrectangle R, formed by the Cartesian product of neighborhoods of each a_i, such that any point r in R whose i^{th} coordinate is not equal to a_i, will provide the initial locations of a system which generates *no* multiple collisions .

Standard Position Theorem 2 : Let S_a be a given system, and p_k a particle which is not responsible for any multiple collisions. Then there is an open interval around a_k of systems equivalent to S_a. The proof follows along the same lines as the First Standard Position Theorem.

It follows from basic continuity, (no jumps in the trajectories) , and the linearity of the equations for locations in terms of earlier locations, that any particle which is not responsible for any multiple collision can be perturbed a tiny distance and still not be responsible for a multiple collision.

Construction: Now, keeping all initial locations fixed except the first, take the particle p_1 and begin moving it to the left. As it does so, the location of the center of gravity will change, but all initial velocities, masses, and relative positions will remain unaltered. Whenever p_1 hits a point at which it is responsible for a multiple collision, a small perturbation to the left will uncouple those collisions. Whenever p_1 is in an interval in which it is not responsible for any multiple collisions it there will be a neighborhood around its location in which all systems generated by its points and the other initial locations, are equivalent.

Thus, fixing an origin which can be,(but cannot remain) the center of gravity, the motion of p_1 to the left decomposes the real line into a series of open intervals in which p_1 is *not* responsible for multiple collisions, each interval being separated by isolated points in which it *is* responsible for multiple collisions. All systems *within* each interval are equivalent.

<.....___a_1__._____.______.___.__....(a_2, a_3, ... aN)___________

The set of multiple collision points may be labelled ,

$$q_1 , q_2 , q_3 ..=\{q_i\}=Q$$

Theorem: **Q has no finite limit point .**

Proof: **If q were a limit point of Q, then it would not be possible to create an interval around q in which there were no locations at which pk would be responsible for a multiple collision, thus contradiction both the First and the Second Standard Position Theorems.**

To prove the General Collision Theorem we proceed by mathematical induction. It is obviously true for N=1 , N=2. For N=3, there are at most 3 collision numbers: (1) The configuration producing a triple collision. (2) The configuration in which p_2 is to the left of its place in (1) ; and (3) The configuration in which p_2 is to the right of this place. See "Trains and Fly" for details.

Theorem: **Assume that the General Collision Theorem is true for systems with 1,2,3 ...N-1 particles. Then there exists a location L, to the left of the system S with initial locations a_2,..., a_N for particles p_2 ...p_N , such that if p_1 is placed at a_1 anywhere**

to the left of L it will *not be* responsible for a multiple collision. In consequence, all such systems will be equivalent.

Proof by induction:

(1) Suppose that the velocity of p_1 is negative,so that it is moving *away* from the other particles. Let the collection of distinct equivalence classes for the system S = p_2 ,...., p_N be designated C(N-1) . Let T be one of these systems.

By the restricted collision theorem T *eventually* takes the form of a system expanding away from the origin in both directions. We will call this phase, after all collisions have occured, the expanding phase. We can therefore place p_1 so far away that the particle p_2 does not collide with it *until that expansion has begun* .

p_2 recoils; it may reverse direction or simply move at a slower speed. If it reverses direction, then in order to hit p_1 a second time, it must reverse direction again. However, to hit p_1 it must be moving *faster* than it was before, as the speed of p_1 has increased owing to the previous collision.

From this and the conservation of energy one sees that:
***Sub-Theorem*: The number of reversing cycles of p_2 between p_1 and p_3 must be finite in number, and a function only of the momenta of p_1 p_2 and p_3 (in the expansion cycle) .**

***Continuing with the proof*: Since we are assuming the theorem to be true for N-1 particles, each time p_2 returns to the system, it has only a finite number of alternative velocities when it reaches p_1 . One can therefore multiply all these numbers to obtain a maximum. Lets say this number is J.**

After p2 has exhausted all of its reversing cycles, it may now continue to strike p1 , but it will be moving only in the forward direction. However, the number of reversing cycles of p3 between p2 and p4 is also finite, and restricted by the conservation of energy.

Proceeding thusly, we eventually place p1 at a place G *so far to the left* **, that there are only a finite number of systems to choose from for each a_1 to the left of G. As a_1 moves leftwards each such system takes up its interval of equivalence, separated from other systems by points at which a_1 is responsible for multiple collisions. Therefore there is a place L , beyond G, such that all systems with a_1 to the left of L are equivalent.**
(*It should be possible to derive a proof based on the fact that the location ratios that determine multiple collisions will eventually be dominated by the absolute value of a_1 so that no multiple collisions can occur after some point L.* **)**

Theorem (General Collision): **Let p_1 , then, be at some location a_1 to the left of L. All systems with p_1 to the left of L are equivalent, so one can gradually move the initial location of p_1 to the right, until it hits the first point at which it is responsible for a multiple collision. One continues to move to the right, generating the pattern of open intervals and multiple collision producing locations described above. This motion generates a finite number E_T of equivalence classes of systems for the particle p_1 and the selected system T.**

C(N) is then obtained by doing the same process for every member T of C(N-1). Since the number of elements of #C(N-1) is finite, the total number of equivalence classes in C(N) is:

$$\#C(N) = E1 + E2 + \ldots + E_{\#C(N-1)} .$$

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