Iso-structuralism and Russell's Paradox

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December 10.04

This paper may appear trite, perhaps, to a Category theorist. Still, I've the impression that it presents something interesting. The motivation for its reflections arises from the desire to make sense of the following "weird object"

$$\boxed{W = \{0, \{0, \{0, \ldots\ldots\ldots}\}^{\infty} \text{ ,}$$

where the curvilinear bracket denotes the "operator" used in forming a set, and the "infinity sign" indicates the collection of right brackets. W cannot be a set as it violates fundamental notions about sets:

1. A set cannot contain itself as an element.
Clearly W ε W
2. A set cannot be an element contained in itself. Ditto
3. The infinite descent prohibition. One sees that W has the property that

..... ε W ε W ε W ε W

This is interesting. It means that W violates both the "membership" taboo , and the "ownership" taboo. W claims to be a member of a club that it cannot belong to, and also claims that, as a club, it enrolls a member that it cannot have.

Here is another object that, as it violates the prohibition against infinite descent, does not qualify as a set, yet does not violate the condition that a set not be a member of itself:

$$\boxed{V = \{0, \{1, \{2, \{3, \ldots\ldots\ldots\ldots}\}^{\infty}$$

It appears that if one can somehow sidestep the prohibition against infinite descent V might qualify as a legitimate entity, one that legitimately "exists" in the universe of mathematical objects. I argue in this paper that V is a legitimate object of *Dual Set Theory* , while W cannot meet the qualifications for existing in either Set Theory or Dual Set Theory.

There is a major technical problem in the publication of this communication; I trust that my audience will bear with me: since it's being sent out in .pdf format , its symbols are restricted to those available to me in that format. Written out by hand, I could easily invert the "epsilon" symbol signifying "inclusion" or "membership to obtain "reverse inclusion", or "ownership". The following notational convention is therefore adopted :

Definition: $\boxed{A \succ B \equiv B \in A}$

Thus the statement that V permits an infinite descent can be notated as :

$$\boxed{V = \{0,\{1,\{2,..\} \succ V_1 = \{1,\{2,\{3,..\} \succ V_2 = \{2,\{3,\{4,..\} \succ ..}$$

(The infinity sign is dropped for convenience)

I. The Isostructuralism Theorem Generalizing Russell's Paradox

We make a distinction between the "membership" and the "content" of a well-defined set. The "membership" of a set S consists of all its elements: $\boxed{S = \{s_1, s_2, s_3,..,,\} = \{s_\alpha\}}$, where α belongs to some index set I.Technicalities viz-a-viz the Axiom of Choice will not concern us here.

The *content* of S consists of all elements e_i for which there are chains of inclusion of the form $\boxed{e_{i_1} \in e_{i_2} \in e_{i_3} \in \ldots e_{i_4} \in S}$, as well as S itself. The prohibition against infinite chains of descent doesn't prevent a chain from being arbitrarily long, even within the same set. There are situations in which one has to specify that the content is restricted to *n^{th} level content* , that is, all chains of length n or less. The set of content elements (*n^{th}-level content, etc*) will be notated as Ω_S. Let us say that a set S is given by S = { A, B } , where A = {1,2,3} , B = {1,2}. One may chose to consider A and B as "elements", that is to say, indecomposable, in which case Ω_S = {A,B,S } . Or one may treat them as sets, in which case Ω_S = {1,2,3, A,B, S} . One could say that the former situation gives the 1st level content. In general, we will just use the word "content" when there is no ambiguity.

Assume therefore that the question of "levels" has been decided in advance. The *structure* of a set S is defined by its *hierarchy of chains of inclusion* . This may be exhibited by writing down the table of all the chains connecting the members of the content set , Ω_S

Example:

S = { a,b,c,d,e} . a ={1,2}, b ={2,5,6}, c ={1,6,f}, d ={2,3,g}, e ={4,5,7}

g = {4,5,6,7, h}, h = {1,2,7}. So:

$$\boxed{\begin{array}{l} 1 \in a \in S; 1 \in c \in S; 2 \in a \in S \\ 2 \in b \in S; 2 \in d \in S; 2 \in f \in c \in S \\ 2 \in h \in g \in d \in S ;\ldots \end{array}}$$

Naturally , S, or Ω_S , or both many have infinitely many members, yet, since each chain must end in S as its right-most element, each chain is of finite length. For example, consider the set:

$$\boxed{\Delta = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\{\{\phi\}\}\}, \ldots.\}}$$

The elements of the content set are

$$\boxed{e_1 = \phi; e_2 = \{\phi\}; e_3 = \{\{\phi\}\}, etc.}$$

Given any positive integer, N, there is a chain

$$\boxed{e_1 \in e_2 \in \ldots. \in e_N \in \Delta}$$

However, although the following formula is correct, it is not a chain, since there is no final term which is a member of Ω_Δ :

$$\boxed{e_1 \in e_2 \in \ldots. \in e_N \in e_{N+1} \in \ldots.}$$

In fact it is enough to require that all chains be finite. If e is any element of Ω_S then , by the prohibition against infinite descent, it can only be a finite number of levels away from S. Therefore, the above diagram can be replaced by the collection of all finite chains related the elements of Δ . The collection of all finite chains connecting elements of Ω_S can be designated C_S .

Two sets A and B are said to have the same *hierarchic structure* , or simply the same *structure* , if there is a 1-1 correspondance κ between the content sets Ω_A and Ω_B , which is also an isomorphism between their chains:

***Definition 1* : Sets A and B are called *isostructural* if there is a 1-1 correspondance $\boxed{\kappa : \Omega_A \leftrightarrow \Omega_B}$ such that**

$$\boxed{\begin{array}{l} [x, y \in \Omega_A \wedge x \in y] \to [k(x) \in k(y)] \\ [u, v \in \Omega_B \wedge u \in v] \to [k^{-1}(u) \in k^{-1}(v)] \end{array}}$$

Definition 2 : A *hieromorphism* between A and B is a *surjective* (many-one) mapping $\boxed{\zeta : \Omega_A \leftrightarrow \Omega_B}$ such that $\boxed{[x, y \in \Omega_A \wedge x \in y] \rightarrow [\zeta(x) \in \zeta(y)]}$

An example of sets A, B between which there is a hieromorphism which is not an isostructuralism is :

$$\boxed{\begin{array}{l} A = \{\{a,d\},\{b,e\},\{c,f\}\} \\ B = \{\{a\}.\{b\},\{c\}\} \\ \Omega_A = a,b,c,d,e,f,\{a,d\},\{b,e\},\{c,f\},A \\ \Omega_B = a,b,c,\{a\},\{b\},\{c\} \\ \zeta : a \rightarrow a; d \rightarrow a; \{a,d\} \rightarrow \{a\} \\ \quad b \rightarrow b; e \rightarrow b; \{b,e\} \rightarrow \{b\} \\ \quad c \rightarrow c; f \rightarrow c; \{c,f\} \rightarrow \{c\} \end{array}}$$

The Iso-Structural Theorem **: No well-defined set S can be iso-structural to one of its proper subsets, R**

Proof : Let s be an element in S which is not in R.

$$\boxed{s \in S - R = S \cap R^C}$$

s exists because R is a proper subset. Assume that there is an isostructuralism from S to R , κ . Then κ carries s into an element r, which is in R : r = κ (s) . By definition of an isostructuralism, then the inclusion $\boxed{s \in S}$ carries over to an inclusion $\boxed{\kappa(s) \in \kappa(S) = T}$. Now T= κ(S) can neither exceed nor be an element of R; because if it includes elements not in R then it is not an isostructuralism from S to R; and if it is an element of R, then there is a chain of inclusions from T to R whose elements

correspond to no inverse mapping . It follows that κ (S) = R and R is an *element* of S.

If R is an element of S, then an infinite descending chain is set up of the form : $\boxed{S \succ \kappa(S) = R \succ \kappa(R) \succ \kappa^2 R) \succ \ldots}$**, which is prohibited in standard set theory . Q.E.D.**

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The Isostructrualism Theorem *extends* the prohibition against sets which include themselves as elements to a more general class, those which include an *isostructural* subset or element. Once more examining the structure of the object V,

$$\boxed{V = \{0,\{1,\{2,\{3,\ldots\ldots\ldots.}\}$$

one realizes that it violates both the isostructuralism theorem and the infinite descent condition, However, it does not contain itself, as W does:

$$\boxed{W = \{0,\{0,\{0,\ldots\ldots\ldots.}$$

One is naturally led to ask if there is some natural generalization ST* of Set Theory (ST) which gives V at least formal existence. By formal existence the following is meant:

The formalism which defines ST* will be consistent if and only if ST itself is consistent.

We think that there is and call our hybrid Dualized Set Theory.

Dualized Set Theory

The relation of inclusion, $\boxed{\text{"}\in\text{"}}$ **, expresses the idea inherent in a "hierarchy" The defining properties of a hierarchy are, in**

every respect, apposite to those which define an equivalence relation:

Hierarchy $\boxed{(\in)}$	**Equivalence Relation (~)**
$\boxed{(1)\neg\exists a \mid a \in a}$	$\boxed{(1)\forall a \mid a \sim a}$
$\boxed{(2)\neg\exists(a,b) \mid a \in b \rightarrow b \in a}$	$\boxed{(2)\forall(a,b) \mid a \sim b \rightarrow b \sim a}$
$\boxed{(3)\neg[a \in b \wedge b \in b \rightarrow a \in c]}$	$\boxed{(3)\forall(a,b,c)\ [a \sim b \wedge b \sim c \rightarrow a \sim c]}$

Note that the transitivity property (3) ***may*** **pertain between some members of a hierarchy. This is important as it allows for membership of elements at different levels.**

The 3 defining conditions of a hierarchy are all satisfied if "membership" is replaced by "ownership", that is, if $\boxed{(\in)}$ **is replaced by** $\boxed{(\succ)}$**.**

Hierarchy (Membership) $\boxed{(\in)}$	**Hierarchy (Ownership)** $\boxed{(\succ)}$
$\boxed{(1)\neg\exists a \mid a \in a}$	$\boxed{(1^*)\neg\exists a \mid a \succ a}$
$\boxed{(2)\neg\exists(a,b) \mid a \in b \rightarrow b \in a}$	$\boxed{(2^*)\neg\exists(a,b) \mid [a \succ b \rightarrow b \succ a]}$
$\boxed{(3)\neg[a \in b \wedge b \in b \rightarrow a \in c]}$	$\boxed{(3^*)\neg\forall(a,b,c)\ [a \succ b \wedge c \succ b \rightarrow a \succ c]}$

If Set Theory were completely dual, the distinction between "element" and "set" could be arbitrarily assigned. Or one could choose the direction of "membership" or "ownership". "Nation" and "Citizen" would be interchangeable concepts, just like "line" and "point" in Projective Geometry. We know in fact that Set

Theory is not completely dual. It would be so if the following two Postulates could be dualized:

$$\boxed{\begin{array}{l} P_1: \exists x \rightarrow \exists\{x\} \\ P_2: \exists S | S = \{s_\alpha\}, \alpha \in \omega \\ \wedge s_0 \in s_2 \in \ldots\ldots \end{array}}$$

The first postulate states that every "existent thing" is somehow "settable", that one of the attributes of "existence" is the possibility of being an element in a set. This is not dualizable. The following postulate does not hold in standard set theory:

$$\boxed{P_1: \exists x \rightarrow \exists u | x = \{u\}}$$

This is not satisfied, for example, by the null set. Even if "nothing" or "nullity" could be ascribed some sort of existence, it could certainly not be thought of a set formed from something else!

The second postulate is a consequence of the Axiom of Infinity. It is not dualizable because of the prohibition against infinite descent. The dual would imply the existence of a set R, every of which *contains* another element that is also either a member of R or in the content of R.

Because the inversion of "membership" , *as a relation* , is completely dual to "membership ", it is possible to extend Set Theory by a formal system in which even "emptiness" has content and infinite descents are allowed. The consistency of this formal system would be pegged to that of Set Theory itself, although it might be difficult to find "real world" examples of entities or concepts as representations of such a system (Modern Physics,

with its enduring inability to define "empty space", may provide such a representation).

In this short communication we limit ourselves to extending Set Theory only so far as to admit certain new objects based on the object V previously notated as :

$$\boxed{V = \{0,\{1,\{2,\{3,\ldots\ldots\ldots\ldots}$$

Let

$$\boxed{V_n = \{n,\{n+1,\{n+2,\ldots\ldots\ldots\ldots}$$

Then, at least in a formal sense, one sees that

$\boxed{V \succ V_1 \succ V_2 \succ \ldots}$**, i.e. an infinite descent.**

We will show that V is dual to a set- representation of ω defined by

$$\boxed{\begin{array}{l} O = \{a_0, a_1, a_2, \ldots a_n, \ldots) \\ a_0 = \phi \\ a_{n+1} = \{a_n\} \\ a_0 \in a_1 \in a_2 \in .. \end{array}}$$

To set up the formal equivalence between V and O, one must construct the dual entity for V, to the "content set" for O. Observe that the content set Ω_O , has exactly the same elements as O, and that the dual-content set ΩV , consisting of all the "sub-objects" in V, also have exactly the same number of "elements" as V. The dualism can be expressed as :$\boxed{\delta : a_n \leftrightarrow V_n}$**.**

Although ω is not an element of O, it is possible to treat O as a *representation* of ω by O and to argue therefore, that every element of O is in the content of ω . In the same way, one can

argue that although ω is not contained by any element of the sequence $\{V_n\}$,it is possible to treat V as a *dual representation* of ω , so that therefore ω is contained *in* every Vn .

Note also the parallelism between the 3 axioms of a hierarchy with respect to both the elements of O and the dual elements of V.

$$\boxed{\begin{array}{l} 1{:}\, a_n \notin a_n \\ 1^*{:}\, \neg V_n \succ V_n \\ \\ 2{:}\, a_n \in a_{n+1} \rightarrow a_{n+1} \notin a_n \\ 2^*{:}\, V_n \succ V_{n+1} \rightarrow \neg V_{n+1} \succ V_n \\ \\ 3{:}\, \neg[a_n \in a_{n+1} \wedge a_{n+1} \in a_{n+2} \rightarrow a_n \in a_{n+2}] \\ 3^*{:}\, \neg[V_n \succ V_{n+1} \wedge V_{n+1} \succ V_{n+2} \rightarrow V_n \succ V_{n+2}] \end{array}}$$

In other words, O and V are isostructural*under* the dualism which exchanges "set" and "element", and "membership" and "ownership". By the isostructural theorem, V cannot be a set. One might call it a "repository".

The object represented by the notation

$$\boxed{W = \{0,\{0,\{0,\ldots\ldots\ldots}$$

is prohibited by both ST and ST*, for one has both:

$$\boxed{W \in W_1 \in W_2 \in \ldots}\text{, and}$$

$$\boxed{W \succ W_1 \succ W_2 \succ \ldots}$$

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A simple model for V in terms of some "real world" phenomenon , is present in biological inheritance.The genetic material I inherit from my parents is a combination of their genetic material and whatever mutations may have occured in their genes.

If we combine their (non-mutated) genes into a common term, G, and their mutations in another term M, then my inheritance H can be represented as

$$H = \{M, G\}$$

Their genetic makeup was a function of both the genes and the mutations of their parents: thus $G = \{M_1, G_1\}$. Clearly the iterative formula is $G_{n-1} = \{M_n, G_n\}$ *The total genetic information* **about myself must separate genetic makeup from genetic mutations at every stage. Note that each mutation acts on all previous mutations contained in the genetic substrate. The appropriate representation of the data relevant to my inheritance is therefore given by**

$$G = \{M_1 , \{ M_2 , \{ ....M_N, \{G_N\}\}\}....\}$$ **, where**

presumably, the creatures whose combined genetic makeup was G_N

were hominids or monkeys. One might also decide to the extend the hierarchy through to fish, yeasts and amoeba, (with the understanding that in a finite world the chain must end at some point) . The finite model is readily extended to an abstract model for inheritance in which characteristics have been transmitted from an infinitely distant past. One might call it the:

*Dual Representation of the Classic Chicken/Egg Paradox.***(!!)**

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