Relative Motion and Reference Frames on 2-manifolds

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I. Introduction

General Relativity argues that gravitational attraction is equivalent to Space-Time curvature. Thus, modifications of the theory can proceed in two ways: one can extend the domain of gravitation or one can generalize the definition of curvature. Alternatives to the current "dark matter" hypothesis allow for reinterpreting "dark matter" as a kind of "dark curvature". Thus, in the basic field equation:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

one might equally well place new terms on the right with the energy-momentum tensor, or with the curvature terms on the left.

In this paper we are asking a different question: if one were to eliminate the force of gravity, the forces of electro-magnetism and inertia remaining, what would be the relativistic behavior of matter, with inertial frames moving along geodesics in a space that is intrinsically curved?

Following the spirit of classical physics, the interactions of material objects, notably collisions, are treated as accidents. The colliding molecules of Statistical Mechanics are neither "attracted" nor "repulsed"; nor are they "bound" in any way by the phenomenon (*which can be interpreted as a kind of force*) of inertia. Unless they happen to be travelling along straight lines on a collision course) their trajectories are assumed to be as independent as wave-fronts, which, by the principle of superposition, are independent under all conditions; that is, they *can* occupy the same place at the same time.

By interpreting inertial interactions in this way, it was possible for Maxwell and Boltzmann to introduce probability and statistics into their models of the foundations of Thermodynamics. Force *fields* such as

gravity and electro-magnetism, which establish causal connections between the movements of independent bits of matter could not be so easily accommodated into a picture of complete randomness (ergodicity).

Gravitation introduces a causal bond. The bond is attractive until the separated pieces of matter collide. Collisions obey the conservation laws of inertia. ***Repulsion by collision thus reunites the causality of gravity with the aleatoric character of collision*****.**

Thus, the collision of the earth with an asteroid 65 million years that killed off the dinosaurs was treated as having been ***caused*** **by the mutual gravitation of both masses, but the rebound of the clouds of dust and rocks that led to the death of the dinosaurs is treated as an effect of inertia, an 'accidental' entanglement of pieces of matter.**

When considering geodesics on surfaces with a non-uniform curvature, pre-existent before the introduction of any material particles, the seemingly clear line of demarcation between causal and accidental behavior fades into a blur.

Let us consider the possibility that our ordinary space has an intrinsic curvature, a combination of topological and metric attributes which do not contribute to any gravitational force, but which effectively "warp" the geodesics of ordinary Special Relativity. This might give the impression of a gravitational force to observers in some reference frames, but not to others. However, being of an "inertial" nature, the seeming attractions between objects would be accidents only.

I am thinking of the special case of geodesic motions on the surface of a sphere. As we know, they do not form a group. If O is an observer at rest "outside" the sphere, and O observes observers P and Q moving at uniform velocities on what to him are distinct geodesic curves, P will not see Q as moving on a geodesic but a polynomial of 4^{th} degree in the tangents of angles on the sphere. In other words, ***O will say that P and Q are moving inertially,*** **with no applied force, but** ***P and Q will see each other as being impelled by some kinds of force. Yet P and Q will each think of themselves as being at rest*** **, in the sense that it is impossible to design any experiment to detect their own motion.**

P and Q will identify the poles of their respective great circles, which are at rest in their reference frames. The poles of P (Q) will be appear to be at rest for P (Q). There may be an object R, on a line from P to its north pole that P claims is at rest relative to itself; but the "objective observer" O outside the sphere will claim that it is moving along a smaller circle, a latitude around the pole, not on a geodesic. For R to persist along this path requires a force, which, presumably, R feels in the form of a constraint, pressure, heat, etc. ***Although P will inform R that it is at rest, R will disagree.***

All of these phenomena arise because geometric lattice of rest frames or reference frames cannot be set up on a sphere. The notion of "relative rest" is associative, and implies a group or at least a semi-group structure. This is equivalent to requiring that velocities add according to some commutative addition law.

The investigation of such manifolds constitutes an essential default background for General Relativity: the subject of ***Special Relativity on Curved Manifolds*** **has something of the status of the** ***Semi-***

Classical treatments of Quantum Theory that seek to model quantum effects by classical means.

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Why should gravity be causal but inertia accidental? Descartes worried about such things, but didn't have the Newtonian tools to deal with them. In a more general sense, the existence of rest or reference frames in nature depends upon the possibility of relaxing the restrictions of causation so that some systems can be allowed to move about independently, or, one might say, interact accidentally. That is why rest frames can serve as the basis of Special Relativity, but are abolished in General Relativity. The notion of being "unable to detect one's motion" loses meaning in the absence of a group structure of mutually observable free trajectories.

This calls for a special comment. Even if one grants the operation of a strong "principle of equivalence" in nature, it only covers those motions along the space-time geodesics carved out by the force of gravity; since it cannot cover (theoretically possible) movements *on non-gravitational or non-geodesic paths, it cannot cover all motions. Now it is not clear to me that geodesity in General Relativity has a group structure, though the "Principle of Co-Variance" seems to imply that it should. If X sees by Y and Z as moving along space-time geodesic paths, will X and Y see each other as moving on geodesic paths? This is not even true on a sphere without gravity, so why should it be true in real space-time?*

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Detecting one's own absolute motion

We will say that an observer O is *"able to construct an experiment to detect his own velocity"* **if:**

(1) *O's velocity is changing, that is, accelerating*. **Since gravity, but not inertia, is being put aside in this discussion, we assume no principle of equivalence.**

In the way of a brief digression, we want to point out a strange peculiarity in the hostility against Copernicus and Galileo in the 17th century by the Catholic cult, with regards to the motion of the Earth.

Even if one agrees that the Earth be motionless in its own reference frame, that of an approximately spherical surface, it can only be so at the equator and the poles, which are the only regions of the surface moving in uniform motion. The motion in all other parts of the planetary surface is non-geodesic, hence detectable by the criterion cited here. Sensitive measurements indeed would have shown that Rome itself was flying apart!

(2) *A changing Gaussian curvature of the space in which O is situated can be measured.* **For example, on a 2-manifold, O can draw a triangle around his feet and measure the area and the angle defect. If this is changing with time, O should be taking this as evidence for an intrinsic motion of some sort. Note that this method is not applicable on surfaces of constant curvature, namely the plane, sphere, cone, cylinder, ruled surfaces, hyperbolic plane, tratrix, etc. It is in fact only on such manifolds that the notion of the undetectability of one's proper motion makes sense.**

(3) *O is moving along a line that is not a geodesic of the surface* **. If the surface is of constant Gaussian curvature, it is possible that O might be moving in such a way, by slowing down and speeding up, so that the**

kinetic energy of motion K= ½ M $|v|^2$ is constant. But then the momentum must be changing, that is to say the distribution of the lengths of the components of v, at each point.

All of these methods for detecting one's motion are local. If local means detect no changes, we will say that O is a *rest observer* . In a smooth manifold, one can extend this to a maximal local reference frame, a construction that may vary with time. Here is an example: Let C be the surface of a cone, and imagine an observer O moving along a generator line (a line through the vertex) at a uniform velocity away from the vertex V. P will be a *local rest observer* . Its *local* reference frame can be built out of the collection of all particles moving up the cone surface on parallel straight lines at the same velocity v on straight lines, or non-generator geodesics. not generators. "New" particles may come into existence at points on the cone as O rises up to their height. These will "disappear" if O reverses direction.

Thus "rest observers" can exist in the absence of "reference frames". A reference frame consisting of "rest observers" will be called a "rest fame"

However, in order for O to define a *global reference frame* , a 4th means for detecting absolute motion is required.

(4) *O observes the motion of a "fixed star"* , that is to say some object which, by virtue of the model being employed, is hypothesized to be immovable. Even Newton employs fixed stars in his system of the world. He needs them to define absolute motion in the case of rotation of a pail of water, that is to say, absolute centrifugal motion.

We will assume that these are the only criteria that allow an observer to identify his own motion. If O passes all these tests, then it can claim to be "unable to conduct an experiment to detect its own motion".

Reference Frames , Rest Observers, Rest Frames

(1) A *local rest observer* O is one that cannot make an experiment to detect its motion through local means (changing curvature, non-geodesic orbit, acceleration).A *global rest observer* has no fixed objects in its observable universe by which it can argue, directly or indirectly, for an intrinsic motion.

Thus, a particle on a cone surface, moving along a generator with a uniform velocity, is a local rest observer, since it can take the motion of the vertex as a signifier of absolute motion.

A particle moving with a uniform velocity on a sphere is a global rest observer.

(2) Reference frames also come in two varieties, local and global. They are composites of observers at relative rest to each other, and need not be either local or global rest observers:

(a.)A local reference frame is defined as follows: Let O be a given observer. Place it at the certain of a ball of some unspecified radius . At any collection of points in this ball it is either theoretically or practically possible to place a clock, ruler, theodelites, etc. which over time record no change in the relative displacements of observers at these points with the initial observer, or with each other. This state can be a temporary or eternal.

Note that the property of being a local reference frame is not transitive: B can be in a local reference frame of A, C in a local reference frame of B, yet C is not in the local reference frame of A.

(i) A particle p moving on a generator on a cone C, away from the vertex V with uniform velocity v, can draw a circle K around itself which, if its boundary does not overlap on the cone, can serve as a local reference frame for all other particles moving up on geodesics segments that are truncated by the boundary of C . If q is a point near the boundary of this circle , one can draw a circle K' around q which will contain points r not in K, that are at rest relative to q but not to p.

(ii) The spherical surface does not admit of reference frames: its inertial motions do not form a group.

(iii) One finds 3 distinct classes of geodesics on a cylindrical surface: generators, helices and loops. All the same the cylinder does admit of reference and rest frame structures. The 3 classes are interchangeable: one can, for example, turn a "loop geodesic motion " into a "generator

geodesic motion " or "helical geodesic motion" by moving the observer along a suitable trajectory at the right velocity.

(b) A *global reference frame* is defined as follows: given any collection of locations observable by O, it is possible to place test particles and measuring instruments which, moving relative to O and to each other, appear to be at rest. One touches here on the difficult matter of covariant descriptions of nature as allowed by General Relativity. For example, if p is a particle in Euclidean 3-space, moving along some erratic, (say fractal) trajectory, it is, in theory, possible to place observer particles at any number of points in the plane, which, moving along congruent fractal paths in parallel to O, appear to be at rest relative to each other. Global reference frames are therefore possible in Euclidean 3-space. This property is transitive, as it depends on a group structure: If p is at rest relative to q and q relative to r, then p is at rest relative to r.

Even Newton observed that if all objects were to speed up with an identical acceleration, all in the same direction, the change would be undetectable. Therefore:

(b.)A local or global *rest frame* is a local (global) reference frame of a *rest observer*.

***A rest frame*, then , is composed of rest observers which remain in isometric relationship to each other . Special Relativity adds a further twist to this definition by demanding that these isometries be established by signals traveling at the speed of light. One may call this situation that of a relativistic rest frame. Rest observers and local frames do not require signals traveling back and forth to establish them, whereas general reference frames appear to cause difficulties; that is to say, by local measurements one might establish that all clocks and particles are moving with the same acceleration in the same direction, while the signals between them will travel at an absolute fixed speed of light.**

Rest frames and reference frames come equipped with clocks theoretically stationed, or "stationable" , at every point that is admitted into the frame. As the example of a uniform motion on the cone away from the vertex shows, reference frames can vary with respect to time,

growing or shrinking as a particle advances or recedes. For this reason it is important that there be a unique "origin" observer O in a reference frame. O has with it a clock K, that does not appear, disappear or alter with time, but beats periodically in a stable fashion with no detectable alteration of period for all relevant situations under discussion.

A brief digression: In discussions of Special Relativity it is always assumed that the "light signals" are immaterial, pure transmitters of information, although we know that a real light ray carries energy from one place to another, hence altering (however minutely) the physical makeup of the body which receives it. In fact it can be argued that, in any realistic portrait of the physical universe, all transfers of information involve the interaction of energetic bodies which are altered by the transaction.

Reference Frames and Relative Motions on Cones

We will limit our discussion to two representative cones. One can use these cases to develop the theory of reference frames and inertial motions on cones in general. These are:

(1) The Cone C_π . If one opens this conic surface and lays it flat, the vertex angle of the infinite sector with be π: i.e., it covers the upper half plane. Its properties are detailed in my paper "Double Intersection Geometries : www.fermentmagazine.org/intersect.pdf . Any two points not on the same generator (line through the vertex) have 2 geodesics between them.

(2) *The Cone* $C_{\pi/2}$. If the cone be opened and laid flat, the vertex angle is π/2: the flattened and opened surface covers the upper right quadrant. For points not on the same generator, one can draw *4 geodesics* between them. All geodesics (the use of the word 'geodesic' refers here only to non-generator straight lines*)* *loop once* : they come in from infinity, loop around and self-intersect, then go back off to infinity. again. All geodesics are similar. Locally their properties are indistinguishable from Euclidean straight lines.

All of the intersection properties, as well as the affine relations of uniform motions along geodesic lines, can be depicted in a modular array of Cartesian graphs, quotiented by the vertex angle.

Thus to depict the properties of geodesics on C_π one makes a cut along a given generator from the vertex. This creates an upper half plane, which can be rotated counter-clockwise to create a *rotated duplicate* that captures the behavior of lines and curves that cross the line L.

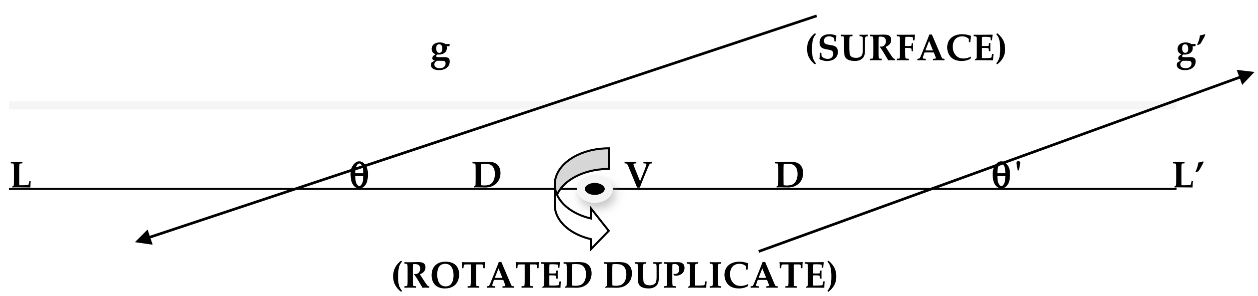


Figure 1

The lines VL and VL′ are *two representations of the same generator line* from V, which has been cut so that the surface can be laid flat. It also serves as the abscissa of the Cartesian coordinates, with the origin at the vertex V.

The geodesic line g crosses L to the lower half plane. Its continuation, g′ is on the other side of the vertex, cutting the abscissa at the same distance D and identical angle θ. Notice that in this geometry geodesics *do not* cross themselves to make a loop. Using this diagram it

is easy to show that there are two geodesics between any two points not on the same generator.

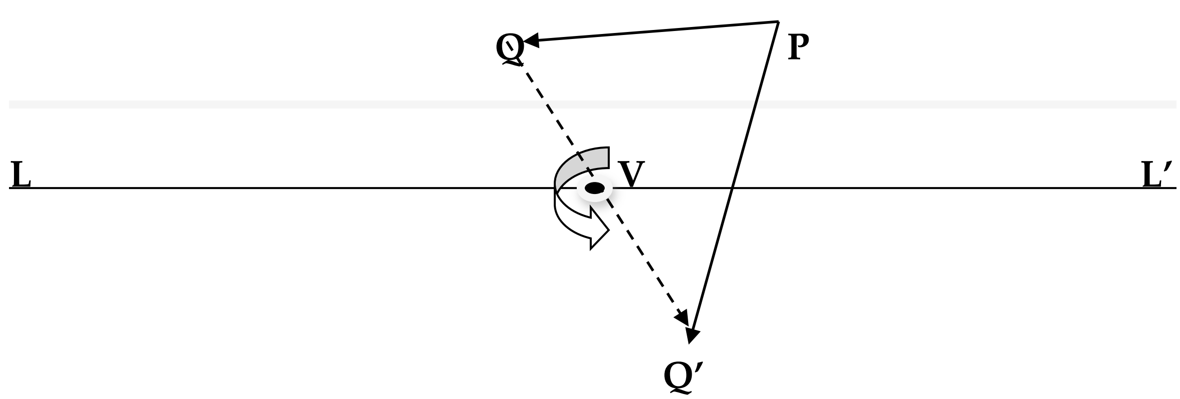
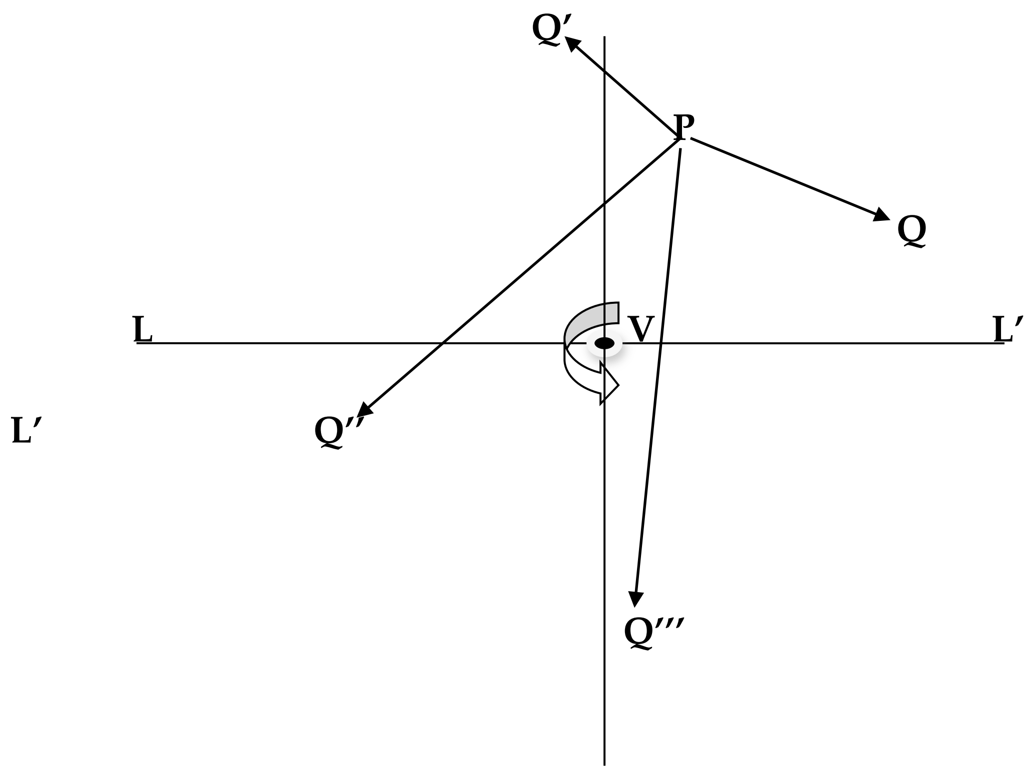


Figure 2

Q and Q′ are the same point. Any other point P not on the line QVQ′ will connect with P in two different ways.

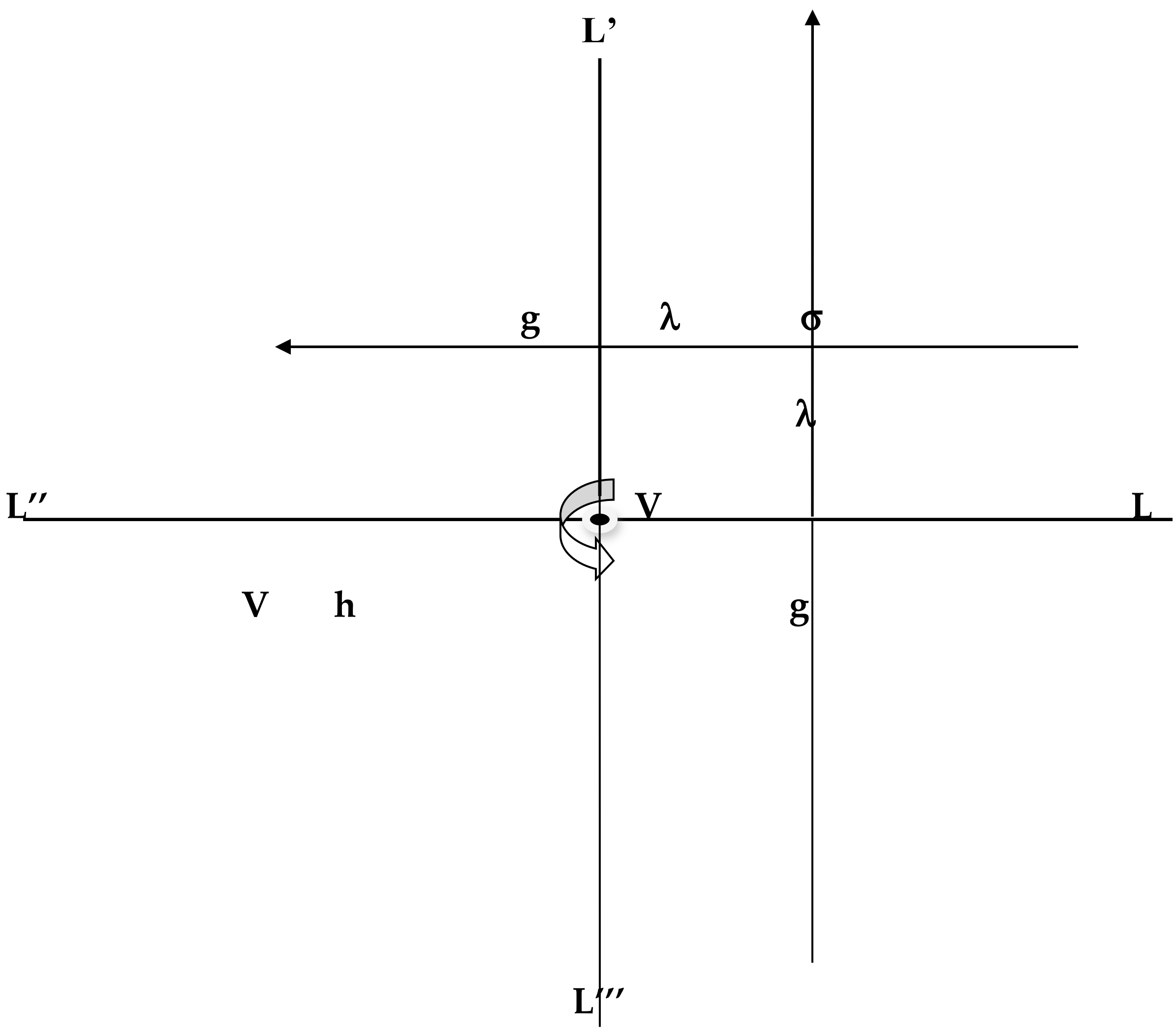
Such a diagram can be constructed for any cone. If the vertex angle divides 2π integrally, the number of sectors will be finite. If the ratio with 2π is rational, the number of sectors is finite, but overlapping. If the ratio is irrational, the number of sectors is infinite, corresponding to geodesics which cross themselves infinitely many times.

For $C_{\pi/2}$ the diagram looks like this:



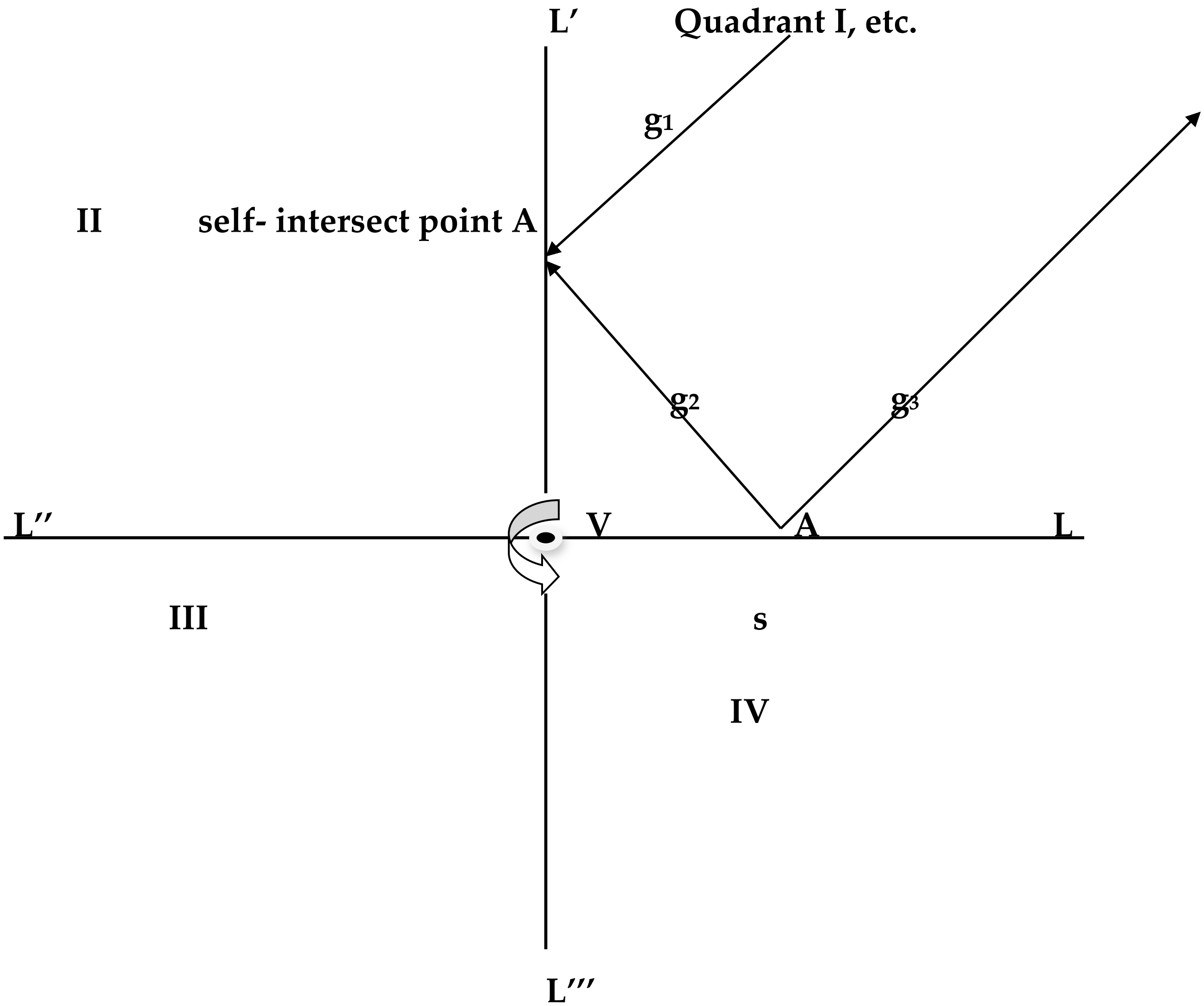
The diagram shows that there are 4 geodesics between points not on the same generator line. All geodesics are similar, with a single looping intersection, symmetrically arranged relative to the vertex and a single generator passing through the intersection point. There are two natural diagrammatic representations:

(1)



The cone is opened along a generator LL′ that is parallel to the geodesic gg′. The two “branches” g, g′ are in fact the same line extended to show the loop λ and the “intersection point” σ

(2)



In this diagram one sees the self –intersecting loop as a line between the two versions of the "cut line" LL′ on the cone. The geodesic g thus has 3 sections, the two branches arriving from and returning to

infinity, and the loop itself. The seeming incorrect arrow heads are actually correct, and indicate this reversal of direction of the flow.

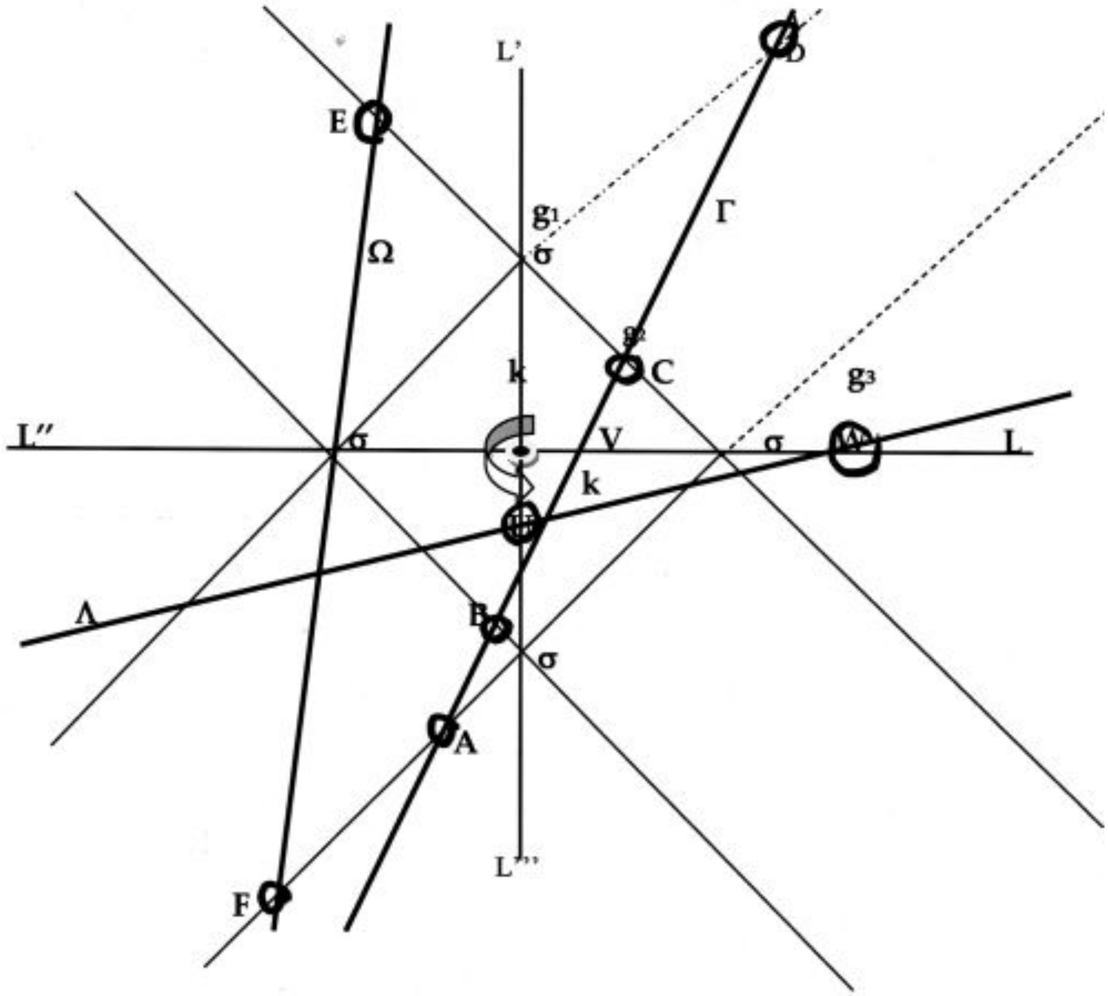
With the use of this diagram, one can easily display all the possible intersection relationships between geodesics and generators. If the self-intersection points of two geodesics coincide, the geodesics coincide. This is easily proven.

Theorem: Let p be any point on the cone, $C_{\pi/2}$. There is a unique geodesic g that has p as the intersection point of its loop. **The proof depends upon two facts, clearly displayed in the above diagram:**

(1) The angle of intersection between the branches is π/2

(2) The geodesic of which p is the intersection point is symmetrically disposed relative to the line between p and the vertex.

Figure 2 may be enriched by including the equivalent representations of the line g in all 4 quadrants. This creates a square around the vertex to represent the loop, and a series of 8 branch lines A line drawn between equivalent points represents a loop:



Note the intersection points A,B,C,D,E, F, U, W. Construct the geodesic g. Open the cone $C_{\pi/2}$ on the line between the vertex and the intersection point σ of g. From the diagram, one sees that the geodesic Γ cuts g in 4 points (A,B,C,D), while the geodesic Ω that goes through σ cuts g in 3 points (E, σ, F). One may therefore treat the intersection point

as a double point. Observe that the geodesic Λ cuts the generator line L=L′=L″= L‴ in 2 points, U and W. 2 generators meet only at the vertex (Or possibly in Heaven). This completes the catalog of intersection options between geodesics and generators.

Reference Frames on C_π and $C_{\pi/2}$

(1) Local Reference Frames on C_π

If one generator line L′L is taken from the Cπ surface, one can consider the remainder to be a local reference frame for any *stationary* point p provided that one excludes all geodesics that cross L′L. To bring in relativistic considerations, one must look at what happens when particle p at some location O is set moving along a geodesic at a uniform velocity v.

There are two classes of geodesic on Cπ, those which pass through the origin and are called generators. When the word “geodesic” is used in the context we will mean a non-generator geodesic, one that does not pass through the origin.

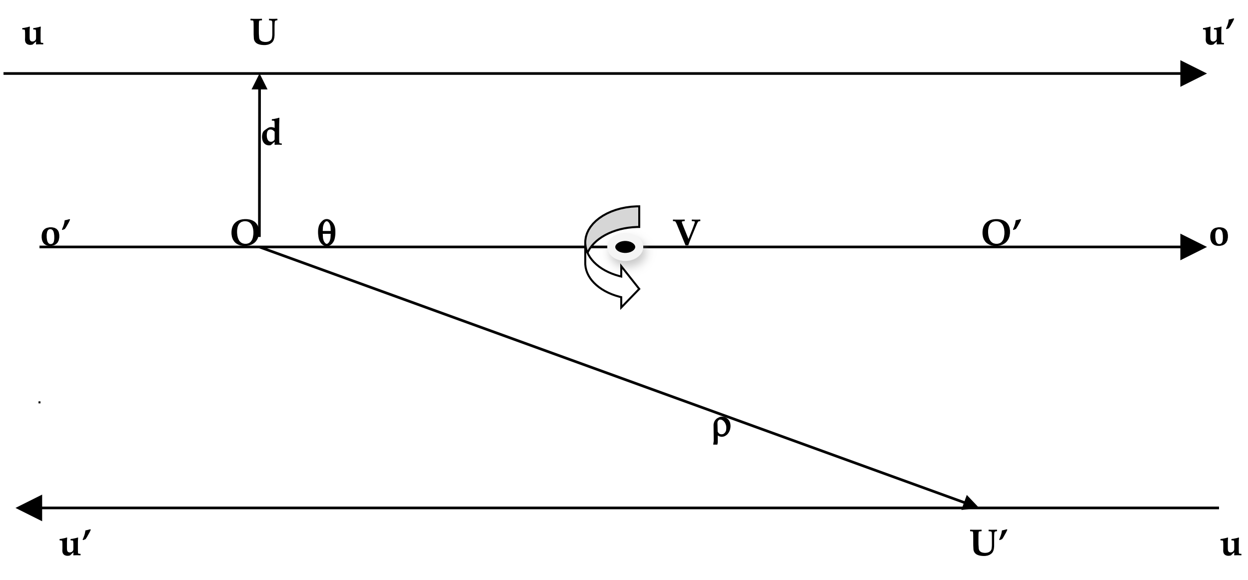
Since the surface is flat, p will be unable to detect its own motion *by local means* , but can detect its motion through reference to V, the vertex, which operates like a "fixed star" in this universe.

Motion along a generator

Specifically, if O moves away from the vertex along a generator, at a uniform velocity, it will see V moving away from it at the velocity v. By making an adjustment for this motion, it will again be able to interpret all uniform motions on C_π as inertial. In particular, all observers or particles moving at a velocity –v on lines going to the vertex will be interpreted as being "at rest" in the absolute frame. The reasoning is not dissimilar to that which we use in choosing to fix the center of gravity in the sun at rest relative to the solar system.

Theorem: If O starts moving away from V, at a velocity v and along the generator line connecting p with V, it can, under the assumption that V is a fixed star, invoke a mathematical *transformation which will provide him with the "absolute" reference frame of the surface of C_π .*

However, there another kind of local reference frame which O can view his own motion directly. The world lines of particles rising parallel to his world line at the same velocity, will appear to be at rest relative to himself, *provided he exchange light signals in one direction only***. In fact, if observer U on line u appears to be parallel to the line o of O by light going in one direction, u will diverge from o as witnessed from the exchange of signals along the complementary geodesic. Open the cone on the line oo′:**



The cone has been "opened up" along the generator line of the motion of the observer O. The motion along the line uu′ has been reversed in the lower half plane. The two lines, from O to U and from O to U′, are the two geodesics. One sees that although the distance and angle between O and U remain identical and perpendicular, therefore parallel, the geodesic between O and U′ as drawn on the lower half plane will be changing in both direction and length as U moves will velocity v along the line u′u.

It is not difficult to show that the apparent shape of the secondary trajectory of U, as seen by O, will be a straight line. In fact we have:

The distance between O and O′ is 2vt = h.

The distance of O to U (or O′ to U′) is fixed at d.

Therefore:

$$d = \rho \sin\theta$$
$$2vt = \rho \cos\theta$$

Thus O sees the other image of U moving along a straight line parallel to OV, his line to the vertex, at a speed double his speed from

the vertex(x coordinate) , and perpendicular (y coordinate to the line to the vertex. He detects no local motion, either in himself, nor in the particles moving at the same speed in lines parallel to the line OV.

Local rest frames on C_π

1.) If an observer O, at particle location p, and uniform velocity v from the vertex, determines by purely local experiments, that it is "at rest", O can then establishe a continuum (actual or theoretical) of other observers and particles "at rest" relative to itself and to each other.

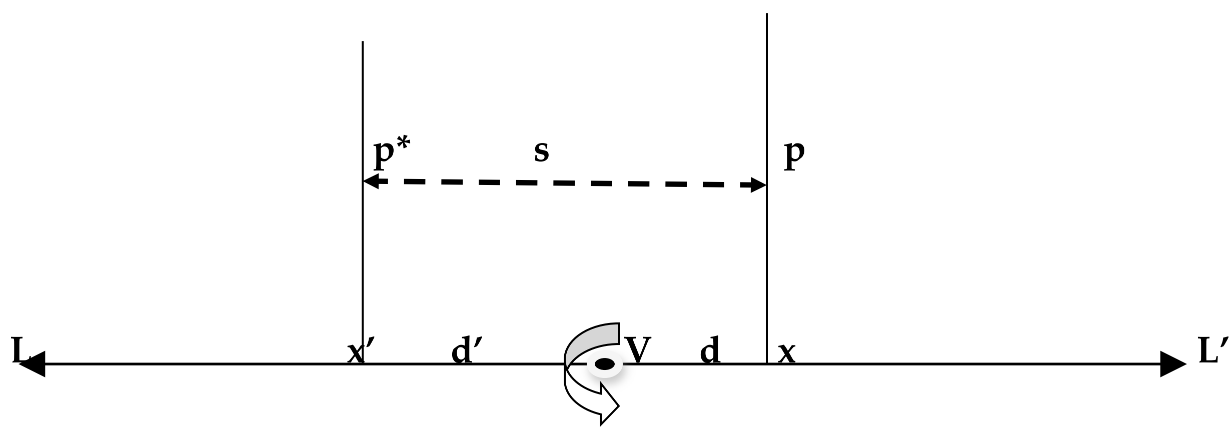
2.) Having constructed this "relative rest frame", one can use it to Describes the appearance of other particles moving in uniform velocity relative to O.

A. In C_π, one can speak of the 'self-observation' of the geodesic along which a particle may be moving, even though there is no local means of determining one's own motion. moving along a non-generator geodesic g, away from V with uniform velocity v.

Let the particle be designated p. Although there are no local means for establishing p's motion, there are two non-local methods:

(i) p can take sightings of the vertex, V

(ii) *Geodesics on C_π are parallel to themselves* : one can therefore observe the motion of a particle on a different branch, thus indirectly establishing one's own motion. Here is the diagram:

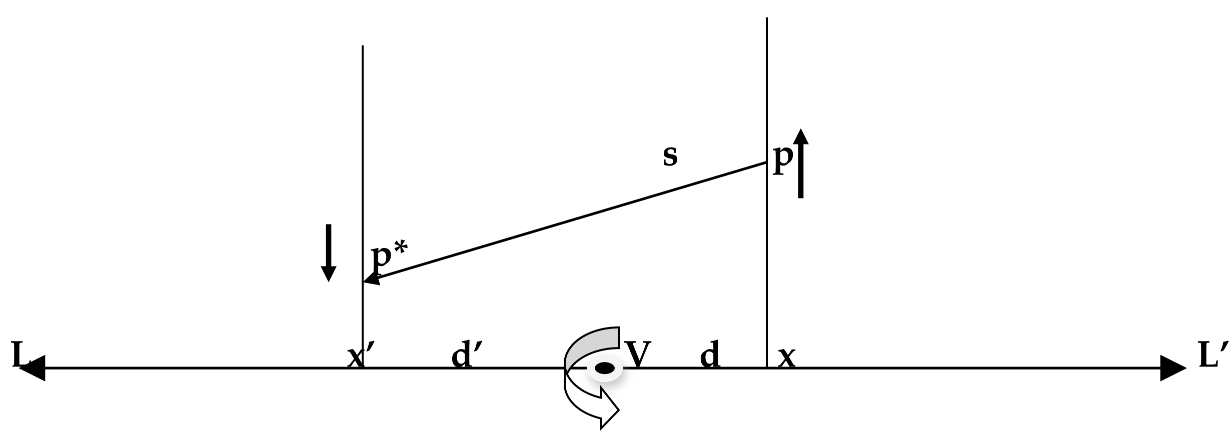


The cone has been opened up on the line LL′ perpendicular to the geodesic notated as p′x=xp . The lines LV and L′V are the same line, as are the lines d′V and dV, but the branches px and p*x′ are different. When the cone is closed, the two branches p*x′ and px coalesce to make a single line. The length d does not contribute to the geodesic but represents the distance to the vertex from p to the vertex V.

Imagine p moving away from V with uniform velocity v. Then if p* is also moving away from V , they will actually be moving in *opposite* directions. However, p will *observe* p* to be at rest by virtue of signals s passed between them.

In C_π there is always a second geodesic between two points not on the same generator. This second geodesic goes down the length of px, over to x′ then up to p*. Under the interpretation of the information from this geodesic, the particles p and p* are moving *away* from each other with velocity 2v.

An interesting situation develops when p and p* are traveling in the same direction, relative to the cone, but opposite directions relative to the branches of the diagram:



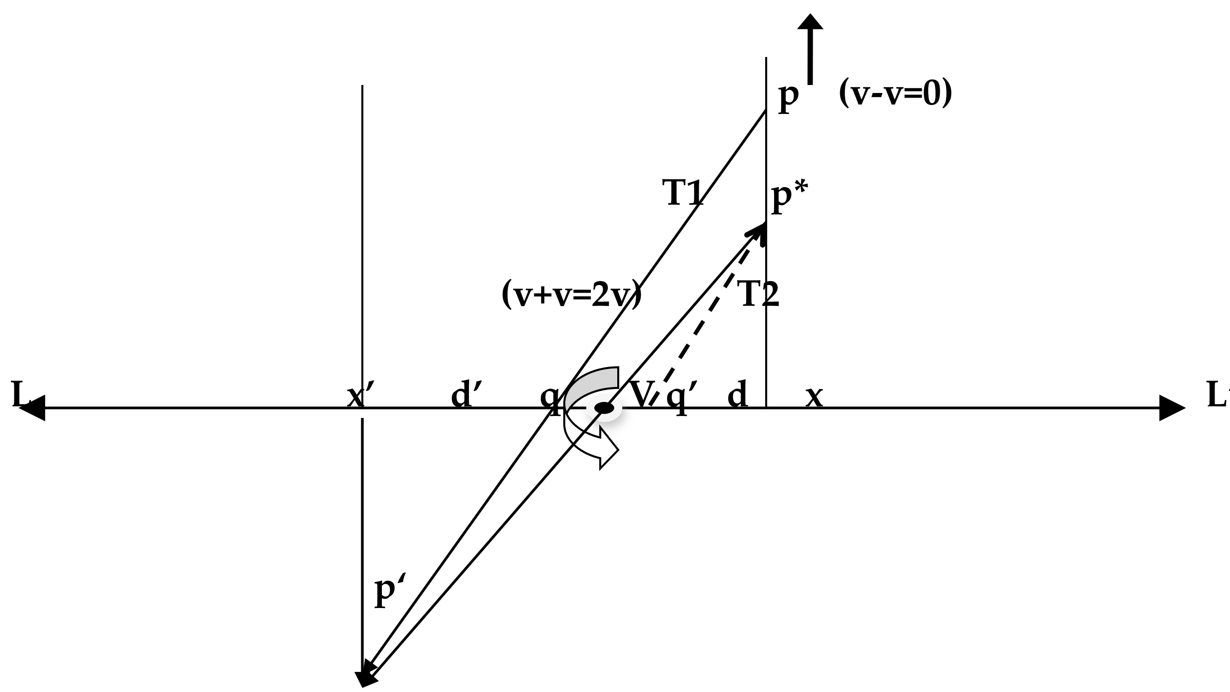
As long as p* is on the left branch and p on the right branch, the two ways of measuring the relative velocity are:

(1) Along the geodesic itself. Then p and p* appear to be at rest.

(2) Measured along the second geodesic,s, p* appears to be moving

away from p on a line with perpendicular distance 2d from p, at velocity 2v.Eventually p and p* will both be on the right branch px

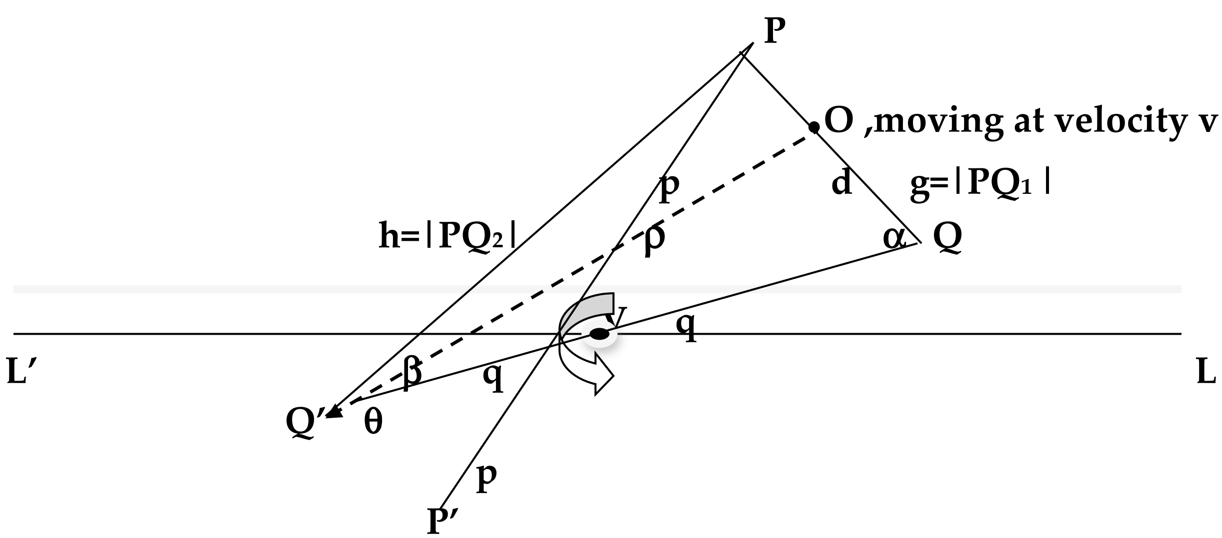
Measured along the line between them, the velocity is 0. To visualize the second geodesic one proceeds as follows:



p′ in the lower half-plane is the same point as p* in the upper. The second geodesic is then drawn from p to p′. This cuts the line LL′ at the point q, which is equivalent to the point q′ on the other side of the vertex. Thus, the total length of this second trajectory from p to p′ is a combination of T_1 and the dotted line T_2. The relative velocity is, once again ,2v.

By examining flows along 2 or more geodesics v, one discovers that certain sets will appear to be in the same reference frame when measured by one set of geodesic signal connections, but in different frames when measured by their complementary signal geodesics. The one sure method of establishing absolute motion is with reference to the vertex.

Observation of a particle at location P, moving along a geodesic to an observer at location Q, at constant velocity v:



The observer "O" is moving from P to Q with a fixed velocity, v, along the geodesic PQ_1, of length g= $|PQ_1|$, Now imagine that, as O is descending, ***it is sending out light signals at each point, along the two geodesics PQ_1 and the line ρ going from the location of O to Q′(=Q.).***

d is the length that O has still to go along PQ_1 to reach Q.

What is Q observing? The line L′L is arbitrary, so one may take the angle α between the baseline Q′Q and the trajectory PQ, as the angle at

which Q sees the approach of O. The angle β may be taken as the initial angle of the light rays going to Q, θ the variable polar angle, and ρ the radius vector of polar coordinates centered in Q. The situation may be summed up as follows: let us say that O is in a "rocket" moving towards Q. Then Q sees two distinct rockets leaving their planet at the same time. The first is moving to Q along the line PQ1, at distance g and angle α. The second is starting from another location, at distance h and angle β, and moving away along a line whose distance from Q′ is the altitude from Q′ to the line PQ. Q′ is of course the same point, seen from a inverted mirror perspective.

The second rocket appears to be moving away from Q, but when the polar angle θ equals 0, there is, once more, a "flash" of a light ray moving from Q to the vertex and back again. This reveals that the secondary trajectory was a mirage, as a single particle can only move along a single trajectory. This is the fundamental distinction between material motion, which is directed on a 1-dimensional path , and electro-

magnetic expansion, which can move in all directions. Note that the Doppler shifts in the light arriving at Q in two directions may differ.

Local reference frames on $C_{\pi/2}$

The basic properties of geodesics on the $C_{\pi/2}$ surface are:

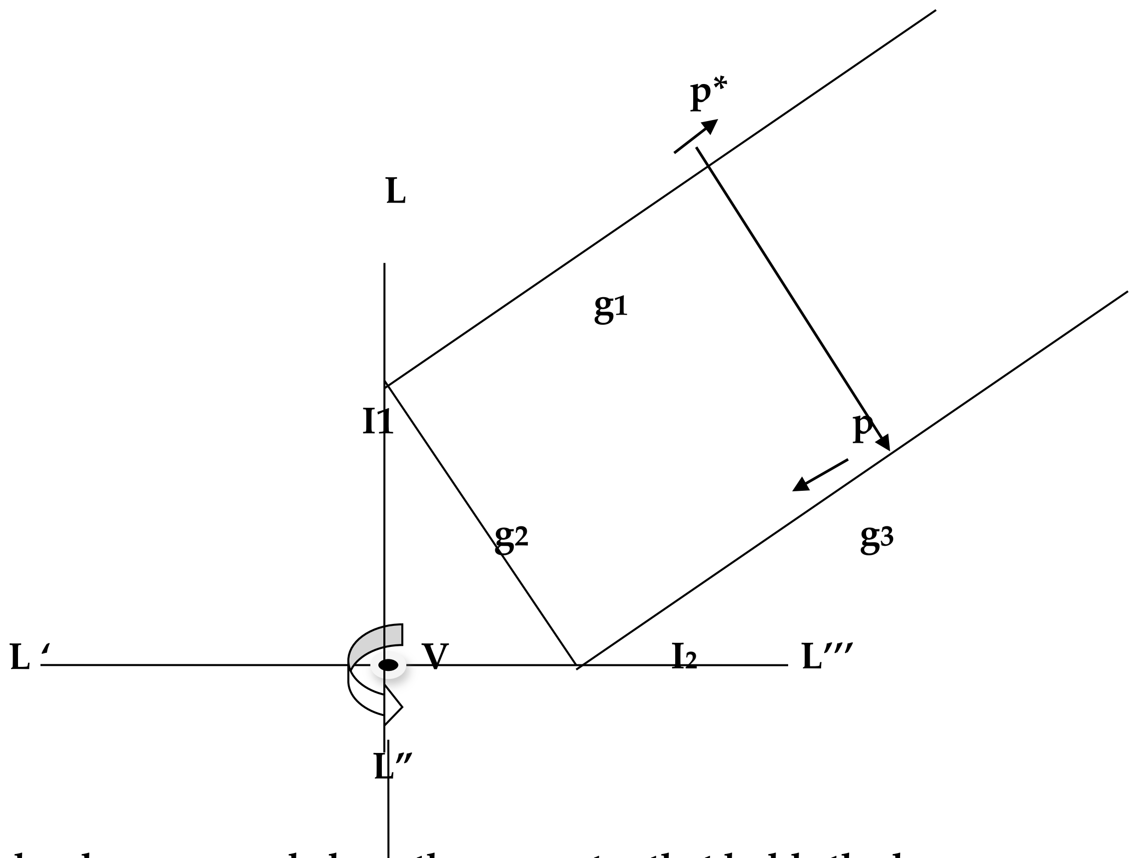
(1) All non-generator geodesics are similar.

(2) Geodesics make a single loop with two branches intersecting at 90 degrees.

(3) The two branches, left and right, are parallel

(4) *Dualism:* **Between two points not on the same generator, there are 4 geodesics. Two non-generator geodesics intersect in 4 points.**

(5) The loop intersection point is sufficient to determine the geodesic.



The cone has been opened along the generator that holds the loop intersection point. Thus, a traveling observer would go down the left branch of g. At the intersection with the axis L’V, it switches over to the other side (*Think of its extension into the second quadrant and the rotation of the chart carrying the second quadrant into the first*), then progresses back to the intersection at L’V, where it once again “jumps” to the intersection at LV and begins to climb the right branch.

Once again we place particles p p′ onto the two branches. p is rising on the left branch, p′ descending on the right branch

***Phase 1*: p, on g_1, sees himself as being at rest, although an observer off the cone sees it as climbing up g1 at a uniform velocity, v. p′, on g3, appears to be descending at a speed of 2v, until it hits the loop.**

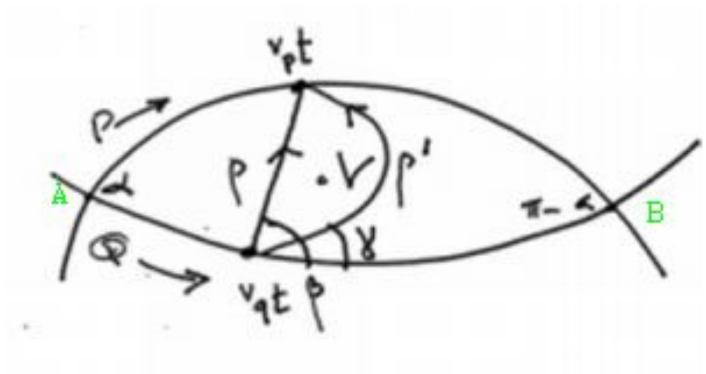
Phase 2: p′ has entered the loop, jumped across, and is now moving back to the rightmost intersection point, at velocity v. According to the measurement on the short geodesic from p to p′, the composite velocity, as seen by p, is $\sqrt{2}$v !

Phase 3: p′ has moved completely to the left branch and is moving behind p at the speed v. Thus the relative difference is 0.

Important note: On $C_{\pi/2}$ 4 geodesics will pass between the points p and p*. Only one of these actually holds the particles in motion; the others change from one moment to the next and transmit information via signals such as light rays.

Relative motion on a bangle in C_π .

Points on C_π not on a generator will have two geodesics connecting them. The corresponding geometric object will ***be a figure with just two vertices*** **, dubbed a "bi-angle" or bangle.**



(Bangle formed by observers P and Q in relative motion.)

The properties of bangles that are needed for this section are the following:

(1) The sum of the vertex angles (on C_π) is π .

(2) The cone vertex V is ***always inside*** **any bangle , though not necessarily at the center**

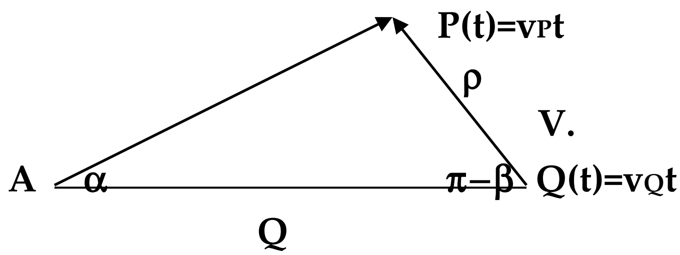
(3) Any geodesic segment connecting points on opposite sides of a bangle (i.e. APB and AQB, must lie entirely inside the bangle.

(4) The two geodesics connecting points on opposite sides of a bangle

also constitute a bangle, and must therefore enclose the cone vertex. The "second geodesic" is (represented as a curve though it is also a straight line on the surface)

P and Q start traveling from A at the same time, P along the upper path, Q along the lower. They also arrive at B at the same time. This means that their velocities must be proportion to the lengths l_P and l_Q. Their geodesics are on a "flat surface", of nul Gaussian curvature. Assuming that Q is unable to detect any intrinsic motion of his own, how does he observe the departure and arrival of P?

Let the velocities of P and Q, as observed from the vertex V, be v_P and v_Q . Then $v_P/v_Q = l_P/l_Q$. At each moment in their passage from A to B, signals between the two voyagers go along two paths. The observer at V claims that the angle between the paths of Q and P at the beginning is α. Since P cannot detect any intrinsic motion by local means, it cannot determine the baseline the baseline AB.



At time t, the distances traveled by P and Q are $v_P t$ and $v_Q t$ respectively.

The radius vector ρ gives the distance and angle of P as it appears to Q. It is a simple matter to calculate the angle β using the law of sines:

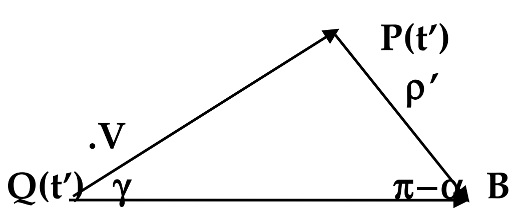
$$v_P/v_Q = h = \sin\beta/\sin(\beta-\alpha)$$

$$\tan\beta = h\sin\alpha/(h\cos\alpha - 1)$$

If the motions are uniform, the ratio h is fixed though the trajectory, and P observes Q moving away at the fixed angle β .

At time t, the distances traveled by P and Q are $v_P t$ and $v_Q t$ respectively.

At re-entry, however, the angle as seen by someone at V is $\pi-\alpha$. The ratio of the lengths traversed and that of distances still to go remains h, but the triangle looks like this:



Notice that V now appears on the left side of the diagram, whereas formerly it was on the right side. This means that the observer has switched from observing the light signals along the initial continuous set of geodesics, to observing the light signals from the complementary set of geodesics. Once again, the law of sines gives:

$$h = \sin\alpha/\sin(\alpha-\gamma) \text{ or } \sin\alpha\cos\gamma - \sin\gamma\cos\alpha = \sin\alpha/h .$$

One can compute tanγ from this though the equation is more complicated than that for tanβ.

At what point does the observer P make the switch from the first set of geodesics to the second? One reasonable solution is to assume that he *switches when the maximum distance of ascent is equal to the distance of descent, i.e. when $\rho = \rho'$* **. This makes sense if one argues that the astronomer on Q will accept those light signals that arrive first as legitimately coming from the observer (object, particle, rocket) P. This means, in effect, that his telescope moves from the inclination towards β to the inclination towards γ. Note that we do not in fact need to require**

that P knows the baseline visible to V. All that is important is that, when the distance of ascent and descent are equal, he moves his telescope through an angle $\phi=\gamma-\alpha$!

The equation is rather cumbersome; if the lengths of the two paths are, respectively, l_P and l_Q, then the equation for the time, at which the distance of ascent is equal to the distance of descent is given by:

$$t=\sqrt{\frac{l_p v_p}{l_q v_q}}\frac{\sqrt{v_p^2+v_q^2+2v_p v_q\cos\alpha}}{\sqrt{v_p^2+v_q^2+2v_p v_q\cos\alpha}+\sqrt{v_p^2+v_q^2-2v_p v_q\cos\alpha}}$$

$$=\tau\frac{\sqrt{h^2+1+2h\cos\alpha}}{\sqrt{h^2+1+2h\cos\alpha}+\sqrt{h^2+1-2h\cos\alpha}}$$

Since $l_p v_p = l_q v_q$ = total transit time τ , the expression at the beginning is given in this form only to maintain its symmetry.

Relative motion along a pair of geodesics in $C_{\pi/2}$.

A. We will apply the Jordan Curve Theorem to conic surfaces. Apart from the Euclidean plane, (the 'uncone' ?) , all other cones hold, in addition to the usual triangles, structures with only two vertex angles connected by straight lines. We call these "biangles", "bangles" for short.

Bangles cannot be constructed in Euclidean 2-space. The very existence of even a single example implies the presence of a non-Euclidean geometry. If A and B are the vertices of the bangle Σ , we make the reasonable assumption that the geodesic segments connecting them do not contain loops.

In combination with the Jordan Curve Theorem, this leads to a theorem I've dubbed the "Bangle Principle" , valid for all cones:

Theorem :

(a) A bangle divides the surface of a cone into an inside and an outside.

Proof: **A bangle is a simply connected, non-intersecting closed loop. Therefore one can apply the Jordan Curve Theorem.**

(b) The vertex V of any bangle must be in its interior.

Proof: **Removing any generator line from the cone transforms its geometry into that of a Euclidean sector. Therefore a bangle must cut every generator at least once. This creates a non-Euclidean loop surrounding the vertex.**

Theorem: Let p be a point on the $C_{\pi/2}$ surface and L_p its associated loop. Then if q is a point in the interior of L_p, the entire loop L_q is in its interior.

Proof: If a segment of the loop of q were to "slip outside" the boundary of L_p, the overlap at the intersection of the boundaries would create a bangle without V in its interior.

Let P and Q be observers moving at uniform on a pair of geodesics g_P and g_Q. The geodesics are assumed to not be generators, and will therefore intersect in 4 points. They also self-intersect at points i_P and i_Q, creating loops. Envisage a trajectory whereby P and Q "descend" from some distant locations on the surface, intersect at point 1, then proceed along their trajectories, making intersections at points 2 and 3, (though not necessarily simultaneously), then do meet simultaneously at point 4 before each goes back out to infinity.

What are the possibilities? These are completely determined by the bangle principle and this list of topological options:

(A)The loop intersection point i_Q lies inside the loop Lp

(i_P lies within L_q)

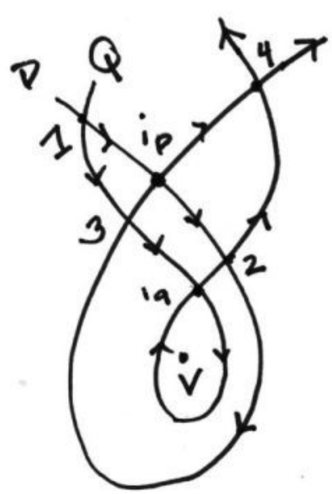
(B) The intersection points i_Q and i_P both lie outside the loops L_P and L_Q respectively

(C) i_Q is on the boundary of the loop L_P (i_P is on the boundary of the loop L_Q)

(D) Each loop intersection point is also on the loop boundary of the other observer.

By application of the bangle principle, one readily shows that each option leads to only one topological configuration:

(A) The loop of Q lies inside the loop of P:



If we follow the trajectories we discover that the sequence of

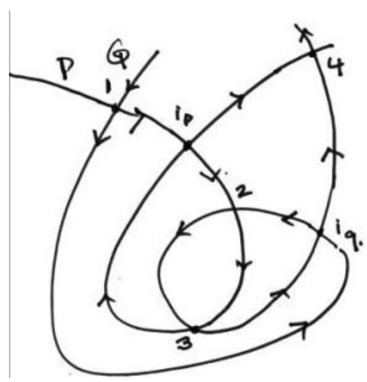
intersections for P and Q respectively, are:

P: 1 i_P 2 3 i_P 4

Q 1 3 i_Q i_Q 2 4

Thus, P and Q cannot come together at all 4 mutual intersection points unless one of them reverses its orbit by a kind of "Ptolemaic epicycle". They can however adjust their velocities to come together at 1 and meet again at 4

(B) The loops intersect:



The sequence of encounters at intersection points is:

P: 1 i_P 2 3 i_P 4

Q: 1 i_Q 2 3 1_Q 4

From this one sees that it *is* possible for P and Q to adjust their

velocities so as to encounter each other at all 4 points 1,2,3,4 . They arrive at different angles and, once again, there must be a discontinuity in the angle of observation between each departure and return.

(C) If i_Q is on the boundary of L_P, then the intersections on both sides of i_Q will coalesce into it, creating a "double intersection point" that coincides with the self-intersection point, i_Q .

(D) The geodesics g_P and g_Q are identical

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