Weakly Infinite Cardinals

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Abstract

A transfinite sequence ,

$$s = s_0; s_1; s_2; \ldots .$$

is proposed. These are defined by the set of relations

$$2^{\sigma_n} = \sigma_{n-1}; n = 1,2,3,\ldots$$

$$2^{\sigma} = K_0$$

After a discussion of the natural arithmetic properties of this series , we restrict our attention for the most part to□□□□for which several models, combinatorial , algebraic, geometric and analytic are proposed .

The combinatorial model is derived from the properties of collections, called "mixets", mixing distinguishable and indistinguishable elements. A bivalent cardinal is defined for them. A sequence of representative mixets is constructed on which a natural extension of the power set operator can be inverted on any cardinal. The inversion on the representative set for K0 produces the cardinal σ *.*

The geometric model for σ *is based on a construction on Hilbert Space called a □ -hedron , (sigmahedron) . Its*

construction raises some questions about the ontological viability of Hilbert Space as an object of geometry. When speaking about a countably infinite dimensional Hilbert Space H , one must recognize that there can be no "internal evidence" distinguishing H from any of its proper countably infinite dimensional linear subspaces.

We call this the :

"Principle of Relativity for Infinite Dimensional Hilbert Space".

This principle of relativity can be expressed in the language of mixets. Plausible arguments show that the cardinal number of the σ*-hedron is indeed* σ *.*

The last model is analytic, utilizing the coefficients of the collection of Fourier series defined by the vertices of the σ*-hedron.*

Introduction

"Mathematics is purely hypothetical; it produces nothing but conditional propositions. Logic, on the contrary , is categorical in its assertions." **- C.S. Peirce**

The cardinal number of the power set P(S) of a finite set S is a simple function of the cardinal number of S .

Let #S = cardinal number of S, #P (S) = cardinal number of P(S) . Then

Theorem I **(Classical): #P (S) =** $2^{\#S}$

Corollary: #P (S) > #S for all finite S, (including the null set, ϕ **) .**

The extensions of this corollary via the Cantor Diagonal Construction, are the foundation from which all of transfinite arithmetic

arises. As it is neither a definition nor a theorem in its own right, the extension of Corollary 1 into transfinite arithmetic should properly be stated as an axiom:

Axiom I : **If T is any infinite set well-defined by the Zermelo-Fraenkel axioms , and P(T) is its power set, then**

$$\#P\ (T) > \#T$$

The customary notation, $\#P\ (T) = 2^{\#T}$, is an arbitrary , not entirely satisfactory convention for infinite sets. The Continuum Hypothesis renders it even more questionable. We will assume the Generalized Continuum Hypothesis in the paper (Jech, pg. 46) because (i) it is not directly relevant to the constructions presented here, and (ii) doing so simplifies the arguments. However, we will not assume that Sierpinsksi's Theorem (GCH ---> AC ; Smullyan and Fitting, pg. 109) applies to the special class of 'pre-countable' transfinite sets that we will be considering.

Other properties of #P for sets, finite or transfinite, are :

(i) If $\#X = \#Y$, then

$$\#P\ (X) = \#P\ (Y)$$

(ii) Conversely,

$$\#P\ (X) = \#P\ (Y) \ \text{---->}\ \#X = \#Y$$

(ii) is perhaps open to question. It is not easy to see how one goes about proving that infinite sets of different cardinalities must produce power sets of different cardinalities. Although a 1-to-1 correspondence ζ : A--->B induces a natural 1-to-1 correspondence ζ *: P(A)--->P(B) , it does not automatically follow that any 1-to-1 correspondence μ* P(A)--->P(B) ***must*** **be invertible into a 1-to-1 correspondence μ: P(A)--->P(B) . However we will assume it here.**

These properties enable us to define a function Γ (n) explicitly on the class of cardinal numbers, C .

If X be a set of cardinal n, P (X) its power set, then Γ(n) = #P(X) = m, where m is independent of the choice of X.

Theorem 2 : #S finite -----> #P (S) finite

#P (S) finite -----> #S finite

 # S infinite -----> P (S) infinite

 # P (S) infinite -----> S infinite

The proof follows from Axiom 1 and because α is always considered to be larger than β when α is infinite and β is finite.

Corollary: " Finitude" and " Infinitude" are invariant under both the power set operation and the inverse power set operation, (defined on the range of P) . Designating the lowest transfinite , $\#Z^+$, by the symbol $\aleph_0$, (Aleph-naught) , a sequence of higher transfinite numbers can be generated from the cardinals of the iterations of the power set operator acting on Z^+ , and on their limit sets. There may exists other processes which generate other transfinite series; we will be looking at one of them in this paper. This series $\aleph_0, \aleph_1, \aleph_2$, ... will be referred to as the *standard sequence* .

***Observation :* The sets in the standard sequence are all either power sets or limit sets of power sets . With the sole exception of $\aleph_0$, their cardinals are either of the form C = Γ(#P(c)), c being the previous cardinal, or $C = \Gamma(\bigcup c; c < C)$. Some subtleties arise from the interplay of cardinals and ordinals. From the perspective of cardinal arithmetic one can say that $\aleph_1 = \#P(\aleph_0)$. From the perspective of ordinal arithmetic ω_1 is the limit of limits of polynomial sequences of the form $\Sigma a_j \nu^j$. If, as**

in Jech's "Set Theory", cardinals are *defined* as limit ordinals (pgs. 25-28;38-39) no problems arise. But as we intend to show here, this identification is an over-simplification.

Theorem III : With the exception of K_0 , all infinite cardinals derive from an iterative or a limit process on other infinite cardinals.

Question: *Where does K_0 come from?*

Making that question meaningful

There does not exist, in standard set theory, a set S with the property that its power set is countably infinite. **This property distinguishes K_0 from the transfinite cardinals that follow it. The next cardinal with the same property is K_ω : we will not be looking at the higher limit cardinals in this paper. The situation invites speculation: might there exist a natural generalization of set theory which allows for the inversion of Γ on K_0 ? Another means for invoking this possibility is to note that all infinite sets with cardinalities greater than K_0 have proper subsets that are also infinite but of lesser cardinality . Now that we have learned, (thanks to the inspired investigations of our colleague, Georg Cantor), that the "Infinite " has a hierarchical structure, there exist neither axiomatic nor intuitive reasons for asserting that it has to have an abrupt starting point at the first transfinite , K_0 .**

Arithmetic Properties of the σ -series

It is a simple matter to demonstrate that extending the standard transfinite sequence with the sequence of weakly infinite cardinals ,

$\{ \sigma_j \}$ is consistent with the algebraic structure of transfinite arithmetic. The more difficult task is that of extending standard set theory itself to include a set with σ as its cardinal number.

Once this is done it will be relatively straightforward, through back reconstruction and iteration on the process $\pi : K_0 \dashrightarrow \sigma$ to construct models for the chain of weakly infinite cardinals, σ_1, σ_2, σ_3, ... An example of the way in which this construction might be carried out is sketched in another section.

An obvious requirement for the weak transfinites is that addition, multiplication, and exponentiation be compatible with transfinite arithmetical logic . I say "logic" rather than "laws" , as the structure of this arithmetic is, somewhat arbitrarily, based on generalizations upon the elementary properties of one-to-one correspondence.

The principles of this logic are :

Let α , β be ordinals

Let N be any finite cardinal (positive integer)

Then:

(i) $\alpha \leq \beta \longrightarrow K_\alpha + K_\beta = K_\beta$

(ii) $\alpha \leq \beta \longrightarrow K_\alpha K_\beta = K_\beta$

(iii) $\alpha \leq \beta \longrightarrow K_\alpha{}^{K_\beta} = K_{\beta+1}$; $K_\beta{}^{K_\alpha} = K_\beta$

(iv) $NK_\alpha = K_\alpha + K_\alpha + ... + K_\alpha$ (N times) $= K_\alpha$

(v) $K_\alpha{}^N = K_\alpha \cdot K_\alpha \cdot \ldots \cdot K_\alpha$ (N times) $= K_\alpha$

(vi) $N^{K_\alpha} = K_\alpha$

Addition and multiplication are commutative, associative and (trivially), distributive. Indeed, any algebraic expression involving trans-finites, as long as they do not appear in the exponents, is equal to the

transfinite of highest index in the expression.

Since the weakly infinite cardinals ought to be "stronger" than the integers, the natural extension of this structure is :

$$j \geq k \rightarrow \sigma_j + \sigma_k = \sigma_j$$

$$\forall j,k(\aleph_k + \sigma_j = \aleph_k)$$

$$j \geq k \rightarrow \sigma_j . \sigma_k = \sigma_j$$

$$\forall j,k(\aleph_k . \sigma_j = \aleph_k)$$

It is easily shown that the initial segment of the standard sequence, (including Z^+ and all the transfinites up to but not including $\aleph_\omega$) , can be consistently extended to include an initial segment of the weakly infinite cardinals, by means of a representation, μ , onto a semi-group acting on the set:

$$Z^+ \oplus Z_0^- \oplus Z_0^+$$

$$Z^+ = 1,2,3,\ldots.,n,\ldots.$$

$$Z_0^+ = 0,1,2,3,\ldots.,n,\ldots$$

$$Z_0^- = \ldots. -n, -(n-1),\ldots. -2, -1, 0$$

This set can also be notated as:

$$A \oplus B \oplus C$$

$$= a_1, a_2, \ldots a_n, \ldots; \ldots b_m, b_{m-1}, \ldots b_{-1}, b_0; c_0, c_1, \ldots, c_k, \ldots$$

The representation then becomes:

$$\mu : Z^+ \rightarrow A : n \rightarrow a_n$$

$$\oplus Z_0^- \rightarrow B : \sigma_m \rightarrow b_m$$

$$\oplus Z_0^+ \rightarrow C : \aleph_k \rightarrow c_k$$

The structure of the semi-group on the letters a , b, and c is given by:

$$a_l + a_h = a_{l+h}$$

$$a_l \cdot a_h = a_{lh}$$

$$2^{a^l} = a_{2^l}$$

$$q \in B \oplus C, then$$

$$qa_j = q$$

$$q + a_j = q$$

$$q \le p \rightarrow q + p = p, qp = p$$

$$2^{b_m} = b_{m\text{-}1}$$

$$2^{b_0} = c_0$$

$$2^{c_k} = c_{k+1}$$

It is self-evident that this semi-group is well-defined.

Infinity, Actual and Potential
Finitism revisited

The sequence S= { σ_j } furnishes us with a new particular solution to the ancient, (Zeno-Aristotle) , antinomy of potential versus actual infinity. This construction:

(i) Eliminates the philosophically dubious assumption that the *limit* of the finite cardinals is $\aleph_0$ (It does make a certain amount of sense, however, to use this terminology for ordinals, defining the first transfinite ordinal, ω , as the limit of the finite ordinals. The concept of a limit enters naturally into any ordinal process.)

(ii) "Actual" infinity can be restricted to the hierarchy of transfinites, $B \oplus C$. "Potential" infinity pertains to statements involving the elements of Z .

It makes sense to us to posit that the infinite cannot be reached via a limit process on the finite. From this perspective, the expression $\frac{\lim f_n}{n \to \infty}$ is not well defined. On the other hand, an expression something like $\lim f_n = f, if |f - f_n| < e, n > N$

***is* well-defined, as are statements such as $\frac{\lim \sin z/z}{z \to 0} = 1$**

, since these involve infinitesimals. Infinitesimals have to do with continuity, the infinite with counting, which are very different ideas. The infinite ought not to be definable directly in terms of any finite process, although some of its attributes may be defined fin terms of what the finite is not.Thus, one may continue to employ the fiction z--> ∞ , as a kind of short-hand for w --> 0 , w = 1/z .

Mixets

The representation of distinct unordered repetitions of identical elements has been considered paradoxical in European philosophy since 100 B.S. [1] Consider the familiar paradigm of Buridan's Ass:

' *Buridan's ass....a hypothetical dilemma in which a person is postulated as presented with two equally attractive and attainable alternatives and thereby loses freedom of choice.* " (Webster's Third International Dictionary, 1981) .

The 14th French philosopher Jean Buridan managed to hold onto

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[1]**Before Socrates.**

good jobs in the academic world, even after William of Ockham placed his works on the Index. Indeed, much of his professional life was wasted in engaging in spite wars with William of Ockham, inventor of the metaphor of "Ockham's Razor" , the elimination of arbitrary or "ad hoc" hypotheses from scientific theories. [2] **The couplet of metaphors *"Ockham's Razor"* and *"Buridan's Ass"* form an antinomy, that of Action/ Inaction, in the sense of Kant.**

Although such objects are not readily picturable they are at the foundations of a good part of all of the hard sciences: mathematics, physics, biology and chemistry. Examples: The equation w = (z- 2)^{k} has a single root, repeated k times. When talking about one of these roots , it makes no sense to refer to its 'place' in the sequence of roots. However, the binomial expansion of this equation provides us with a set of coefficients $c_j = 2^k \binom{k}{j} = 2^k k!/j!(k-j)!$ **which are in general distinct , and come with a natural ordering provided by the exponents of the developed equation. Thus, finite sequences of indistinguishable quantities can serve as the basis for finite, or even infinite, ordered sequences of distinguished elements. Among these we identify several kinds:**

(i) *Totally ordered sequences.* The elements may be identical or distinguished , but ordinally arranged, as with the set of the coefficients of the polynomial

$$y = x^n + x^{n-1} + \ldots + x + 1$$

(ii) *Sets of distinguished elements which cannot be ordered.* One

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[2]**William of Ockham must have been careful to avoid a too rigorous application of his razor so as not to be burned at the stake for atheism!**

may call these "dual", or "adjoint" sets. The prime example of this phenomenon is the couple √-1 = (i. -i). The assignment of the minus sign is arbitrary. There cannot, in theory, be any reason for stating that one of these two roots has any claim to either the plus or the minus sign. As we know , this is not true of the pair, 1, -1 , in so far as 1x1 = (-1) x (-1) = 1 indicates an essential asymmetry between them.

(iii) *Sets of distinguished elements, each accompanied by a (potentially infinite) list of unique or exceptional characteristics.* **These may be ordered, partially ordered, or unordered. This description applies certainly to the integers, 0,1,2,3,.... each one of which appears to abide on a different planet, but it can also apply to the something like the set of all bounded real functions on the interval [-1,+1] to which no direct scheme of total ordering can be applied. (All indirect schemes depend on one's commitment to the Axiom of Choice.)**

Definition: A *mixet* **shall be a finite or infinite mixture of distinguished, and undistinguished elements. Another way of stating this is to say that a mixet consists of distinguished elements and their multiplicities. Q = (a,a,b,a,c,b,b, d) is a mixet. In certain instances the ordering is important, but in general we shall be concerned with unordered mixets, so that Q can also be written as (a,a,a,b,b,b,c,d).**

Presentations

Consider mixets of the form M = (a,a,a,a,a) . It may or may not be reaching to the outer limits of casuistry to suggest that an Axiom of Choice may be required even for such sets - particularly in those situations in which the content of the anonymous entry , "a", is unknown and can be only determined through an act of choosing .

A philosophical philanthropist tells you that there are five exactly identical gold pieces in a box. You're invited to reach inside the box, feel around without looking , and pull one of them out. You do so, retrieving a valuable coin worth $1,000.

You can keep the gold piece he says , on the condition that you can tell him which of the five pieces you've chosen! You argue that there can't be any way of doing so because, by hypothesis, the pieces are all absolutely identical. He replies: " How is it, therefore, that you were able to select just one of them and none of the others?"

The argument goes back and forth. Finally he announces to you that you will be allowed to keep the gold piece, provided you help him in the solution of this philosophical dilemma, which has kept him awake for several months! A few weeks later you return with an Axiom. Your benefactor is satisfied and lets you keep the gold piece.

What is your Axiom? :

Axiom of Choice for Mixets (finite or in finite):

A mixet S is not well-defined unless an ordinal for S is implied in its definition.

In this particular case the presentation consisted of the way in which the coins were placed inside the box. ***The box, which is basically a reference frame, bestows a unique identity on each coin where none existed before. Take away the box and it will be impossible to make a selection of even one of the coins.***

Definition: *A mixet S is " presented" when its definition asserts (with or without constructibility) , the existence of an ordinal γ of the same cardinality as S , together with a 1-to-1 correspondence between γ and the elements of S.*

Example: Again consider the equation $w = (z-2)^5$. This has five roots, all of them " 2 " . We can create a presentation of this root mixet by forming the derivatives of w. Since $w' = 5(w-2)^4$, we can argue that the *first* root of w is the one that disappears from the root mixet of w' . Clearly, for a finite mixet, if there is a systematic way of distinguishing just a single element in each sub-mixet , (essentially a 'choice function') one will obtain a presentation of the entire mixet through induction. For infinite mixets one needs Zermelo's Well-Ordering Theorem.

All presentations of a finite mixet are equivalent. There is a natural isomorphism between the ordinals associated with all the permutations of a (presented) finite mixet. One may make a further distinction between mixets whose presentation ordinal can be constructed, and those for which there may be at most an existence proof for this ordinal. The former may be called 'presented' sets (mixets) , the later 'presentable' sets (mixets) .

Example: The set of computable real numbers C is not recursively enumerable, yet it is known to be countable. C, therefore, is 'presentable' but cannot be 'presented'.

The paradigm for finite presentable mixets which we will be employing in this paper, is that of the vertices of the ν - hedron T , in n-1-dimensional space. (ν being the Greek letter for n . Thus 'tetrahedron' in 3-space , 'quintahedron' in 4-space, etc.) ,

The set of vertices V_T , of the ν-hedron T , is *presented* whenever T is positioned relative to a frame of reference. In the absence of any frame of reference, V_T is *unpresented* , but then it is still *presentable* by our above definition and is well defined as a mixet.

This point is in need of further clarification. Relative to any reference frame in n-space, the vertices of the corresponding ν-hedron are certainly distinguishable. Given one set of vertex specifications (v_1, v_2 , ...v_{n+1}) , one may, by a combination of rotations and reflections, produce another representation ($v_{\pi(1)}$, $v_{\pi(2)}$, ...$v_{\pi(n+1)}$) , where π is any permutation on n+1 indices. If we eliminate the reference frame and try to speak of the intrinsic properties of the ν-hedron , then we can say that all of its vertices are *n-fold indistinguishable* , meaning that there is no property of any subset of k vertices , $k \leq n$, which is not also present in any other subset of k vertices of V_T .

Bivalent Cardinals

Let B be any mixet:

Definitions : The *internal cardinal* ${}^iB$, is defined as the number of classes of distinguished elements in B.

The *external cardinal* , ${}^eB$, is the total number of elements of B, counting multiplicities.

The *bivalent cardinal* , or simply *cardinal* , of B, is defined as

$$\#B = ({}^iB , {}^eB)$$

Examples:

(1) S = (a,a,a,b,c)

${}^iS = 3$, ${}^eS = 5$, #S = (3.5)

(2) R = (a,a,b,b,c,c)

${}^iR = 3$, ${}^eR = 6$, #R = (3,6)

(3) $T = (a,b,a,b,a,b, ...)$. Lacking a presentation for T, we can say nothing about the external cardinal, but the internal cardinal is given by $^{i}T = 2$

(4) $U = (a,b,a,a,b,a,a,a,b,a,a,a,a,b,....................)$

In the case the mixet has a built-in presentation. We have

$$^{i}U = 2 \ , \ ^{e}U = \aleph_0 \ , \#U = (2, \aleph_0)$$

This definition of a bivalent cardinal for mixets will be sufficient for the arguments in this paper. [3]

The Power Set Operator On Mixets

Definition: **If M is a mixet, then we define P (M) , the power set of M , as a collection of all the distinguished subsets of M, (including the null set)**

Example: **Let S = (a,a,a,b,c). Then P (S) = { ϕ , {a} , {b}. {c} , {a,a} , {a,a,a} , {b,c} , {a,b}, {a,c} , {a,a,b} , {a,a,c} , {a,a,a,b}, {a,a,a,c}, {a,b,c}, (a,a,b,c}, {a,a,a,b,c})**

This definition of the power set of M coincides with the usual definition of the power set when M is a set.

The cardinal of the power set of a mixet can be any integer:

(1) U = (a,a,b) ; P (U) = (ϕ, {a}, {b}, {a,a}, {a,b}, {a,a,b})

$$\# P (U) = 6$$

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[3]**For finite, unordered mixets, one can construct a univalent cardinal which gives more information. Suppose that a finite mixet K is composed of elements $a_1 , a_2 , ... a_q$, with multiplicities $m_1 , m_2 , ..., m_q$. We can then assume that K is so arranged that $m_1 \leq m_2 \leq, ...,\leq m_l$. Define the cardinal ^{n}K as the composite product $= 2^{m1} . 3^{m2} . .....p_l^{mq}$, where p_q is the q^{th} prime number. ^{n}K will be unique for a given distribution of distinguished and undistinguished elements. Then $^{i}K = q$; $^{e}K = (1+m_1) (1+m_2) . .....(1+m_q) = \varphi \ (^{n}K)$, where φ is the Euler φ -function.**

(2) V = (a,a,b,b,c,c) ; P (V) = (ϕ , {a} , {b}, {c},......)

#P (V) = 28

In general , if the multiplicities of the elements of a mixet M, are n_1 , n_2 , ...n_k , then # P (M) = (1+n_1) (1+n_2) ....(1+n_k) + 1.

Homogeneous Mixets

Let $A_n = \underbrace{(a, a, ...a)}_{n}\phi^{-1}$, (with n = 0 for the null set). These will be called 'homogeneous mixets ". The collection **C** of all of these for finite n can be enlarged to include the (presented) mixet A_{K_n} . In general we see that the inner cardinal of a homogenous mixet is ${}^{i}A_n = 1$, the outer cardinal is ${}^{e}A_n = n$, while the cardinal of the power set is #**P**(An) = n+1,

Because of our way of defining the power set operator, **P**, there is , associated with **C** , the set of its power sets, designated
S = { **P**(An) = { Z_n } ; $P(A_{K_0}) = Z_0^+ = (0,1,2,3,.....)$

Taking $C \oplus S$ as our universe, we see that:

(i) The cardinals of the power sets of the elements of C can be any positive integer.

(ii) The power set operator, P, can be inverted from any set of S back to C .

(iii) Z is the power set of A_{K_0}

(iv) The cardinal of Z is K_0 .

We therefore assign, to the set A_{K_0} *, the cardinal* σ. This mixet, which we call the σ-mixet , shares properties both of the singleton {1} , and of Z.

The σ - hedron

Introduction:

Let K be any set of cardinality c. Zermelo's well-ordering theorem says that there exists an ordinal of cardinality c. It can be argued that this does not mean that K can be well-ordered. In order to say that K itself can be well-ordered, one must assert that *any* arbitrary process of selection *must* terminate in *some* ordinal of cardinal c, without any way of knowing which ordinal that will be. Indeed, *knowing* which ordinal one will end up with means that K must have been pre-counted, which is circular reasoning. If one cannot say which ordinal the process will terminate in, how can one say that the process must terminate?

Geometry is the study of distinguishing relations between indiscernibles. The prime characteristic of space is its homogeneity. This is not problematic when the number of dimensions is finite; yet owing to the fact that in a countably infinite dimensional Hilbert space, a rotation can be equivalent to the addition of a new dimension, one must allow for the existence of certain 'pre-countable' infinite sets, such as the collection of vertices of the σ-hedron. Indeed, the term " pre-countable transfinites" may turn out to be more suitable to the description of the series { σ_j } , than the term 'weakly infinite cardinals' used in this paper.

The σ-hedron is a K_0-simplex constructed in a given Hilbert space, which is then cut free of external reference frames. This object is countable, by construction. Yet any counting process will fail to cover all possible vertex collections

. It may be the case that the σ-hedron provides a simple model for the independence of the Axiom of Choice from the rest of Set Theory.

Reference Frame Independent Simplexes in Finite Euclidean Spaces

Let T^3 be a regular tetrahedron in 3-space, considered intrinsically in the absence of reference frames . T^3 is given sequentially to the members of a board of examiners . Each examiner takes T^3 into an isolation cell, (thereby assigning it a reference frame) . After completing their investigations each of them writes up a report which is handed in to the office of the project manager. Here the data is assembled and analyzed. The final result is a document issued in the name of the collectivity.

Among its conclusions one finds that there can exist no way of knowing if the order in which the vertices were inspected by one of the examiners is the same, or different, from that of the others. There are 24 different ways of ordering the set of vertices but no way of knowing which of them was used. [4] **Only with the tetrahedron right in front of them, is there a way of comparing their systems of labeling.**

There was still quite a lot that they could agree on.

(1) Each examiner counted 4 vertices, in the order "1" , "2", "3", "4" . Both the cardinal and the ordinal of the vertex set were 4.

(2) The same intrinsic solid geometry of the tetrahedron is deduced independently by each examiner.

(3) Each maintains that their count, exhausted the set of vertices.

The result is quite general, and can be extended to ν - hedra in any finite n-dimensional space ($\nu = n+1$) : both the cardinal, and the ordinal,

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[4] **A few of the examiners did get into arguments with others who, like them, insisted that *their* labeling method was the correct one.**

of the n-hedron is ν ;the associated labeling process exhausts the set of vertices; the object has an unambiguous internal geometric structure.

This situation changes dramatically when we move to the countably infinite Hilbert Space, H_ω . The object under examination, which we call the σ-hedron , or Δ , has K_0 vertices - meaning that, unlike the situation in finite n-space, its' vertices can be put into 1-1 correspondence with the axes of the reference frame. This observation leads to a chain of unforeseen consequences.

Once again, each examiner in turn disappears with Δ into his isolation cell for an indefinite period of time , studies it thoroughly and writes his report. Now agreement can now be maintained on only some of the previous conclusions :

(1) Each examiner counted the same number of vertices.

(2) The geometry appears to be the same when developed by each examiner. However,

(2) There exists no way of telling whether *even one* of them exhausted the full set of vertices!

It is possible, for example, that the vertices counted by Examiner I were *all* different from those counted by Examiner II. The causes of this are non-trivial: each time an examiner moves from one vertex to the next, he must make an arbitrary leap into a new dimension. *Since the number of dimensions is (countably) infinite, there is no way of showing how the path of one examiner differs from the path of any other*.

We go over the ground in a slightly different fashion, with only two independent examiners, X and Y. The project manager sees both of them at work, but they cannot always see one another. X labels the

vertices of his σ - hedron with the letters U_1 ,U_2 , U_3 ,...... When X has finished, he hands it to Y . Y goes back to his cell and, ***using a method identical to that used by X*** **, labels the vertices W_1 , W_2 , W_3 , ...**

The project manager, who sees everything, realizes that, purely by accident, they happened to have worked in such a fashion that, for all k, $W_k = V_{2k}$. ***Although X and Y are convinced they've examined the same object, Y's σ - hedron is properly contained within X's.***

Under the guidance of the manager, they count vertices together, giving a new series V_1 ,V_2 , V_3 .... Satisfied with their labors they prepare to go home, but the manager stops them at the door. He intends to show them that, no matter how carefully the count is done, it is necessarily incomplete.

Parametrizing the σ-hedron

Within a predetermined Hilbert Space reference frame, the σ-hedron can be built from the ground up, One starts with an equilateral triangle in 2-space, then adds faces and hyperfaces. Let T^n be an n-space ν-hedron with edges of length "1" . Embed T^n in a fixed n+1-space , locate its centroid and erect an altitude h_{n+1} from this point. h_{n+1} can be extended to a point V_{n+1} which is at a distance of "1" from all the other vertices of T^n . Working in this fashion we construct a sequence of vertices:

$$V_0 = (0, 0, 0, 0, \ldots\ldots\ldots\ldots\ldots)$$
$$V_1 = (1, 0, 0, 0, \ldots\ldots\ldots\ldots\ldots\ldots)$$
$$V_2 = (1/2, \sqrt{3}/2, 0, 0, \ldots\ldots.)$$
$$V_3 = (1/2, \sqrt{3}/6, \sqrt{2/3}, 0, 0, \ldots)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$V_n = (x_n^2, x_n^3, x_n^4, \ldots\ldots., x_n^{n+1}, 0, 0, 0..)$$

This construction has the following properties:

(i) For $n \geq 2$, the first n-2 terms of V_n are identical to the first n-2 terms of V_{n-1}.

(ii) For all $i \neq j$, the distance $| V_i - V_j | = 1$

(iii) The length $|V_i| = 1$ for all $i \neq 0$

(iv) The sequence $\{x_n^n\}$ converges to $1/\sqrt{2}$

Let $u_n = v_n - (0, 0, 0, \ldots x_n^n) = (x_n^1, x_n^2, x_n^{n-1}, 0, 0, 0, \ldots)$. **This set of vectors converges to a limit vector, $V_\omega = (1/2, \sqrt{3}/6, 1/\sqrt{24}, \ldots\ldots\ldots.)$**

V_ω has the following properties:

(a) $|V_\omega| = 1/2$

(b) $|V_j - V_\omega| = 1/2$, for all finite j

(c) V_ω is the only point in this Hilbert Space with the above properties. One might be tempted to conclude from this that our σ -hedron is complete: V_ω is the only possible candidate for another σ-hedron vertex, and its' length is half what it needs to be. .

However : **let \$ designate the shift operator. It moves the vertex q= (p_1, p_2, p_3,.....) in H_ω to the point \$q= (0, p_1, p_2, p_3,....). Under the of \$,**

the σ- hedron vertices V_j are moved to $V_j' = (0, V_j)$. In particular, the vertex V_0 remains fixed. <u>$ therefore acts like a rotation on Δ</u> , transforming it into a new σ-hedron Δ' with the same intrinsic relations. Indeed, Δ and Δ' are congruent, but Δ' has a new vertex:

$V^* = (1/\sqrt{2}, V_\omega)$!

***Where did V* come from*? It must have been sitting in another Hilbert Space $H\omega'$ embedding $H\omega$. When we transform $H\omega'$ back into $H\omega$ via the inverse shift operator, (which can be interpreted either as the reverse rotation, or the projection of $H\omega'$ onto $H\omega$), V* dissapears.**

We now return to the story of the independent examiners. After waiting, (with infinite patience) , for them to finish, the project manager points out that if they had rotated Δ a little bit, they would have discovered V*. Everyone goes back to the laboratory, sandwiches V* in somewhere, and begin relabeling. But of course there is no guarantee that we will not neglect other vertices V, V***, and so forth, including some of those from the previous counts.**

Principle of Relativity for Euclidean K_0-Space (Hilbert Space) :

" The Hilbert space **$H\omega$** *is formally indistinguishable from any of its infinite dimensional subspaces. It is intrinsically impossible to devise a test for detecting any feature of a Hilbert space that cannot also be found in any one of its infinite-dimensional sub-spaces. In particular, it is impossible to determine if the space* **$H\omega$** *is or is not a subspace of some larger Hilbert space : an arbitrary leap can always be made into a new dimension. "*

Corollary : The expression $\forall H_{\varpi}$ (*All* of Hilbert Space) is *meta-geometric* ; it is not logically well-formed in the language of geometry

Corollary (Not news) : In the absence of a pre-determined reference frame, there exists no complete orthonormal basis for H_{ω} . Letting F_p stand for the collection of all periodic functions f : R ---> R, then an orthonormal basis for a representation space can be considered complete only relative to that sub region of F_p to which it has been Taylored (sic!) . For example, the Fourier algebra of functions in $\mathscr{L}_2$ $[-\pi,+\pi]$ has as its basis the collection of functions B = { cosnx, sinnx} . B is included in the class $\mathscr{L}_2$ $[-2\pi,+2\pi]$ of all functions represented by the basis B* = { cosnx/2, sinnx/2 }

The vertex set of the σ - hedron will be assigned the cardinality σ , the first weakly infinite cardinal. The cardinality of the power set of a σ–hedron, Δ , is thus the number of *distinguishable* n-simplexes. This is clearly k_0 . We have produced a geometric model for weak cardinal arithmetic.

Defining models for σ_1, σ_2 , σ_3 ,......

This section is only heuristic:

Let U be the set { 0.1} , V the singleton set {0} and W the singleton set { 1 } . Let ζ be a 1-to-1 correspondence from V to W . [5]

The power set of U is P(U) = (x_1 , x_2 , x_3 , x_4) , where $x_1 = \phi$, x_2= {0} , x_3 = {1} , x_4 = {0,1} . The Boolean algebra of union, intersection and complement induces a natural lattice structure over P (U) , which we designate M. The corresponding lattices on V and W can be designated A and B .

[5] Who says that mathematicians have no sense of humor!

The correspondence ζ induces a lattice-isomorphism between A and B , ρ . We wish to extend ζ and ρ to mappings

$$\hat{\varsigma}, \hat{\rho}$$

$$\hat{\rho} : M \to B$$

$$\hat{\varsigma} : U \to W$$

These mappings are extensions of ζ and ρ .All elements of M which are also elements in A are sent into their corresponding element in B . However, elements which are in M but not in A are sent to the null set , ϕ . The mapping $\hat{r}$ therefore induces by back construction the mapping $\hat{Z}$, which sends the element of U which is also in V into the element " 0 " , and the element of U which is not in V into an abstract entity which we shall write as " * " . * is nothing more than a formal symbol with the property that {*} = ϕ . * is perhaps the " content of the null set" . Likewise , the null set can be interpreted as the "power set" of *, which therefore functions as a kind of 'pre-set' . Then * may be defined "implicitly" by means of the diagram:

$$\begin{array}{ccc} P(W) & \xrightarrow{\hat{\rho}} & \phi \\ P \uparrow & & \uparrow P \\ W & \xrightarrow[\hat{\varsigma}]{} & * \end{array}$$

Let δ = (*,*,*,......,*,......) countably many times. δ has the same relationship to σ that σ has to $\aleph_0$. The following postulate seems reasonable : ***Sets consisting of finitely many copies of * are identical to the null set .***

Thus ϕ = (*) = (*,*) = (*,*,*) = ... Under this assumption we can

conclude that the power set P (δ) = { ϕ , ϕ, ϕ, ..., ϕ , ... } . The cardinal of this set is σ . One may, in similar fashion, construct a series of weakly infinite cardinals with the formal property that $s_{n-1} = 2^{s_n}$.

All of this might be interpreted as so much 'symbol mysticism' , which in some sense it is. It may also be understood as a legitimate extension of Zermelo-Fraenkel set theory, consistent with the axioms, representing an original solution to the antinomy of actual versus potential infinity .The assumption that the "infinite" somehow springs *directly* out of the "finite " can easily be dispensed with . The countable sets whose elements and equicardinal sub-sets become distinguishable only when placed within a "box" or appropriate reference frame, provide the essential counter-example.

Families of Orthonormal Functions

Let T = { cosnx, sinnx } be the basis of some Hilbert Space Hω . Let T* be any proper countable subclass of T. By the *Principle of Relativity for Hilbert Spaces* , the space spanned by T is internally indistinguishable from that spanned by T*. This sets up a natural isometry between the respective functional spaces $\mathcal{L}_2$ (T) and $\mathcal{L}_2$ (T*) .

In particular, let T* = T^2 be the collection of functions { cos2nx, sin2nx } , with $\mathcal{L}_2$ (T^2) as the corresponding function space. There are two ways of interpreting the relationship between $\mathcal{L}_2$ (T) and $\mathcal{L}_2$ (T^2) :

(i) One can say that the length of the periods of the functions of $\mathcal{L}_2$ (T^2) are half those of $\mathcal{L}_2$ (T) ;

(ii) One can say that the functions of $\mathcal{L}_2$ (T) are the same as those of $\mathcal{L}_2$ (T^2) , relative to a different orthonormal basis.

By the first interpretation, we stay inside the original Hilbert space and interpret $\mathcal{L}_2(T^2)$ as with the sub-space of functions of period π. By the second interpretation we develop two kinds of Fourier expansion for the functions of $\mathcal{L}_2(T^2)$, over the bases T and T^2 respectively.

This suggests a more satisfactory way of defining the $\mathcal{L}_2$ norm: One usually writes

$$< f,g >= \frac{1}{\pi}\left(\int_0^{2\pi} fgdx\right)$$

We suggest the generalization :

$$< f,g >= \lim_{L\to\infty} \frac{2}{L}\left(\int_0^{L} fgdx\right)$$

Observe that T is unaltered by this new definition. At the same time, the general class of summable functions is now enlarged to include all periodic and almost periodic functions of a finite number of independent, non-commensurable periods. It is then possible to discuss the rotations of the σ-hedron independent of all reference frames.

Returning to the table on page 20, we can express the coordinates of the n^{th} sigmahedral vertex as

$$V_n = (t_1 , t_2 , \ldots\ldots, t_{n-2} , q_n , p_n , 0 ,0 , 0 , \ldots\ldots)$$

The t_j 's represent the growing sequence of fixed terms, while q_n and p_n are unique to V_n . By induction one may show that, for all k > 2:

$$t_k = \frac{2\left(\sum_{1}^{k-1} t_j^2\right) - 1}{2\sqrt{1 - \sum_{1}^{k-1} t_j^2}}$$

$$q_k = \frac{p_{k-1}}{k}$$

$$p_k = \sqrt{1 - \sum_{1}^{k-1} t_j^2}$$

In line with the previous discussion, the application of the shift operator $ produces another σ-hedron including the new vertex

$V^* = (1/\sqrt{2}, V_\omega)$.

In $\mathscr{L}2$ (T) , shifting all the coefficients forward eliminates the constant term. The set of functions G associated to the vertexes of the shifted σ-hedron in Hilbert Space is therefore:

$$g_1(x) = \sin x$$

$$g_2 = \sin x/2 + \sqrt{3}\cos x/2$$

$$g_3 = \sin x/2 + \sqrt{3}\cos x/6 + \sqrt{2}\sin 2x/\sqrt{3}$$

....

$$g_W = \sin x/2 + \sqrt{3}\cos x/6 + \sin 2x/\sqrt{24} + \ldots.$$

With the addition of a constant term, $1/\sqrt{2}$, to g_ω , the forward shift has created a new orthonormal family G*, with the additional member,

$g^* = 1/\sqrt{2} + g_\omega$

G* can in turn be interpreted as an orthonormal basis for the

Hilbert Space $H^*\omega$: the trigonometric functions {cosnx, sinnx} can all be expressed as linear combinations of them. One could therefore build a new σ- hedron on the new basis. The result is a collection of functions $\Lambda = \{ \lambda_j (x)\}$, with a new limit function, $\lambda^*(x) = 1/\sqrt{2} + \lambda_\omega(x)$ outside the space spanned by the g's.

Unless restrictions are placed on the rotations in $H\omega$, which is the same as saying that one begins with a predetermined reference frame, one cannot "count" the vertices of Δ in the usual fashion. Yet whenever a frame is added, its cardinal comes out to be . We therefore assign the 'pre-countable' transfinite of σ to Δ .

The question remains whether the weak cardinal of the limit mixet A_ω = (a,a,a,a,...) is really the same as that of the vertex collection of the σ-hedron. We argue that it is: A_ω is assumed to be presentable, meaning that there exists, in theory, a reference frame, box, or some other kind of presentation with respect to which all of its elements become distinguishable. That it shares this property with the vertex set of the σ-hedron is our motivation for assigning it the same weakly infinite cardinal, σ .

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